Disjoint Boundary-Boundary Paths in Critical Circular Planar Networks

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Abstract

This paper explores some properties of critical circular planar networks. The main theorem, which builds upon the other theorem and lemmas, is that two disjoint edges in a critical circular planar network have disjoint boundary-boundary paths going through them. In the process of getting to the main theorem it is shown that every critical circular planar graph with an interior vertex has at least three boundary-boundary edges and spikes. As a corollary to the main theorem it is shown that in a critical circular planar network every interior vertex has three disjoint paths to the boundary.

1 Introduction

This paper uses the definitions of circular planar graphs and medial graphs from [1]. The following additional definitions concerning medial graphs will be used in this paper.

A boundary triangle is a triangle formed by two crossed geodesics and a piece of the boundary. The sides which are bounded by half-geodesics are called legs, and the side bounded by the boundary is called the base.

An interior triangle is a triangle formed by three geodesics, each of which intersects the other two.

An empty boundary triangle is a boundary triangle which is not intersected by any geodesics.

A geodesic that intersects both legs of a boundary triangle is called an external geodesic and a geodesic that originates at the base of a boundary triangle and intersects one leg is called an internal geodesic.

A boundary triangle that has no external geodesics is called closed and a boundary triangle with at least one external geodesic is called open.

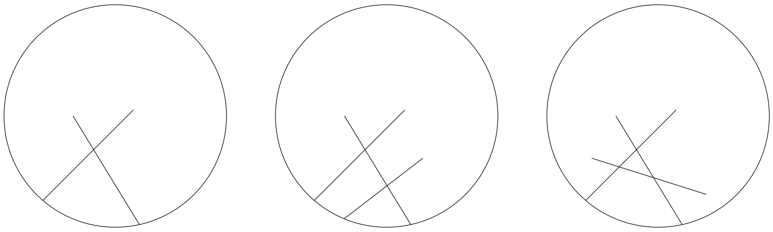

Figure 1: Some definitions illustrated

The following definitions concerning graphs will be used in this paper.

Two edges are considered disjoint if they do not have a vertex in common.

A path of length n is a sequence of vertices and edges $v_1 e_1 v_2 e_2 ... e_{n-1} v_n$, where e_i is an edge connecting v_i and v_{i+1}.

Two paths P_1 and P_2 are disjoint if they have no vertex in common.

A boundary-boundary path is a path of with it's first and last vertices in the boundary.

A simple boundary-boundary path is a boundary-boundary path that has no loops and visits the boundary only at the first and last vertices.

Boundary triangle T_2 is a boundary subtriangle of boundary triangle T_1 if T_2 lies entirely inside T_1. A boundary triangle is a boundary subtriangle of itself.

Boundary triangle T_2 is a proper boundary subtriangle of boundary triangle T_1 if T_2 is a boundary subtriangle of T_1 and T_2 is smaller than T_1.

A minimal boundary triangle is a boundary triangle which has no proper boundary subtriangles.

Two boundary triangles are disjoint if they have no edge or vertex in common.

2 Spikes and Boundary-Boundary Edges

Lemma 1 *If a boundary triangle is closed then it has an empty subtriangle.*

Proof If the boundary triangle is empty then the Lemma holds. Otherwise, at least one of the legs, call it L, has a geodesic intersecting it. Call the geodesic which intersects L closest to the base of the boundary triangle g. g must terminate at the base because the boundary triangle is closed. Consider the boundary triangle formed by L, g, and the base of the original boundary triangle. This boundary triangle, call it t, is also closed because there are no intersecting geodesics in the leg that is formed from L, so no geodesic can intersect both legs. The same argument can now be applied recursively and it must end with an empty boundary triangle because at each step the boundary triangle in question is smaller and the graph is finite. Since each boundary triangle is a subtriangle of the previous, the final empty boundary triangle is a subtriangle of the original.

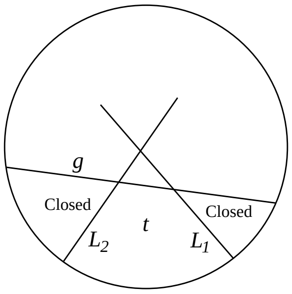

Figure 2: Lemma 2

Lemma 2 *If a minimal boundary triangle t is open then one of its exterior geodesics creates two closed boundary triangles adjacent to the original boundary triangle.*

Proof First, note that since the boundary triangle is minimal it has no interior geodesics, for if it did then it would have a proper boundary subtriangle and thus would not be minimal.

Let L_1 and L_2 be the legs of the minimal boundary triangle. Let g be the external geodesic that crosses L_1 closest to the base. The boundary triangle formed by L_1 and g is clearly closed, since the edge formed by L_1 has no intersecting geodesics, by construction, so there can be no external geodesics. The boundary triangle formed by L_2 and g could have geodesics intersecting both legs. Assume it has an external geodesic. Since this boundary triangle shares L_2 with t, the geodesic enters t. It cannot terminate in t since t has no interior geodesics. It also cannot leave through L_1 because it would either form a lens with g or intersect L_1 closer to the base than g, which it does not, by construction. So there can be no external geodesic and thus both boundary triangles formed from g and t are closed.

Theorem 1 *If G is a critical circular planar graph with at least one interior vertex then the sum of the number of spikes and the number of boundary-boundary edges is at least three.*

Proof This theorem does not require the graph to be connected, but all that is required for the theorem to be true is one connected component with an interior vertex, so in this proof let G refer to such a connected component.

An empty boundary triangle is either a spike or a a boundary-boundary edge, so it only needs to be shown that the medial graph of a critical circular planar graph, in other words a lensless medial graph, contains at least three empty boundary triangles for the theorem to be true.

There are two cases to consider: (1) medial graphs which have an interior triangle and (2) those that don't.

1. Consider a medial graph which does have an interior triangle. Three disjoint boundary triangles are created by the three geodesics which form the triangle. Find a minimal boundary subtriangle for each of these three boundary triangles. In Figure 3 it is assumed that the three original disjoint boundary triangles are minimal for simplicity, but this is not necessary. If all three are closed then by Lemma 1 each has an empty boundary subtriangle, and thus each is an empty boundary triangle, so the theorem holds (See figure 3 (a)). Now assume that one of them, call it t_1, is open, so it has at least one external geodesic.

   By Lemma 2 there exists a geodesic g_1 which creates a closed boundary triangle on each side of t_1. Notice that g_1 can intersect at most one of the other minimal boundary triangles since the medial graph is lensless. Call one of the unintersected boundary triangles t_2. If t_2 is closed then there are three disjoint closed boundary triangles and thus three empty boundary triangles, by Lemma 1 (See fig 3 (b)).

   If t_2 is open then by Lemma 2 there exists a geodesic g_2 which which forms a closed boundary triangle on each side of t_2. Since the graph is lensless at least one of these closed boundary triangles is disjoint from the boundary triangles formed by g_1, so again there are three disjoint closed boundary triangles and thus three empty boundary triangles, by Lemma 1. (See figure 3 (c)).

2. If there are no interior triangles then every boundary triangle is closed, for if there was an open boundary triangle then the legs of the triangle and one of its external geodesics would form an interior triangle.

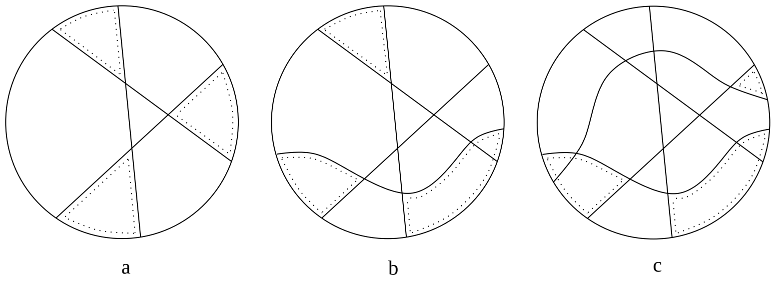

Figure 3: Three cases for Theorem 1, case 1, with closed triangles outlined

Consequently, all that is needed for the theorem to be true is three disjoint boundary triangles.

If there is an interior vertex then the medial graph has an interior polygon with at least four edges (three edges is handles in case 1). Let a, b, c, and d be the geodesics which form four consecutive edges in this polygon. Since there are no interior triangles, a does not intersect c and b does not intersect d, so the four geodesics must be configured as in Figure 4, where d follows one of the two paths indicated. Three disjoint triangles, t_1, t_2, and t_3 are formed, so the theorem is true.

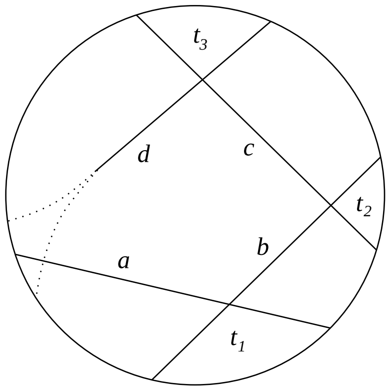

Figure 4: Theorem 1, case 2

3 Disjoint Paths

Theorem 2 *If G is a critical circular planar graph and e_1 and e_2 are disjoint edges in G then there exist two disjoint simple boundary-boundary paths P_1 and P_2 which contain e_1 and e_2, respectively.*

Proof This proof relies on the fact that removal of boundary-boundary edges and promotion of spikes in a critical graph maintains criticality, which is proved in [1].

If G has no interior nodes then e_1 and e_2 are themselves disjoint simple boundary-boundary paths, so there is nothing to prove. Assume there is at least one interior vertex.

Let G' be G with all boundary-boundary edges (excepting e_1 and e_2) removed. Claim: The theorem holds for G if and only if the theorem holds for G'. If the paths exist in G then, since they are simple, they could not have used one of the deleted edges, so the same paths still work. If the paths exist in G' then since adding boundary-boundary edges doesn't change anything, the same paths work in G'. Furthermore, note that G' is also critical.

By Theorem 1 the sum of boundary-boundary edges and spikes is at least three, and G' has no boundary-boundary edges which are not e_1 or e_2, so there must be at least one spike which is neither e_1 nor e_2. Let G'' be G' with such a spike promoted. Claim: The theorem holds for G'' if and only if the theorem holds for G'. Assume the paths exist in G'. There are 3 cases (see Figure 5).

1. Neither path uses the spike's internal vertex. In this case the same paths will satisfy the theorem in G''.

2. One of the paths uses the spike. In this case the path minus the deleted spike will work.

3. Neither path uses the spike but one of the paths uses the internal vertex of the spike. In this case one of the paths visits the boundary twice in G'', so if only half of the path is used then the theorem holds.

Now assume the paths exist in G''. If neither path uses the vertex that used to be a spike then both paths are still valid in G''. If one of the paths does use the vertex then it must end there since the vertex is on the boundary and the path is simple. A new path in G' that satisfies the theorem can be formed by simply appending the spike to the path.

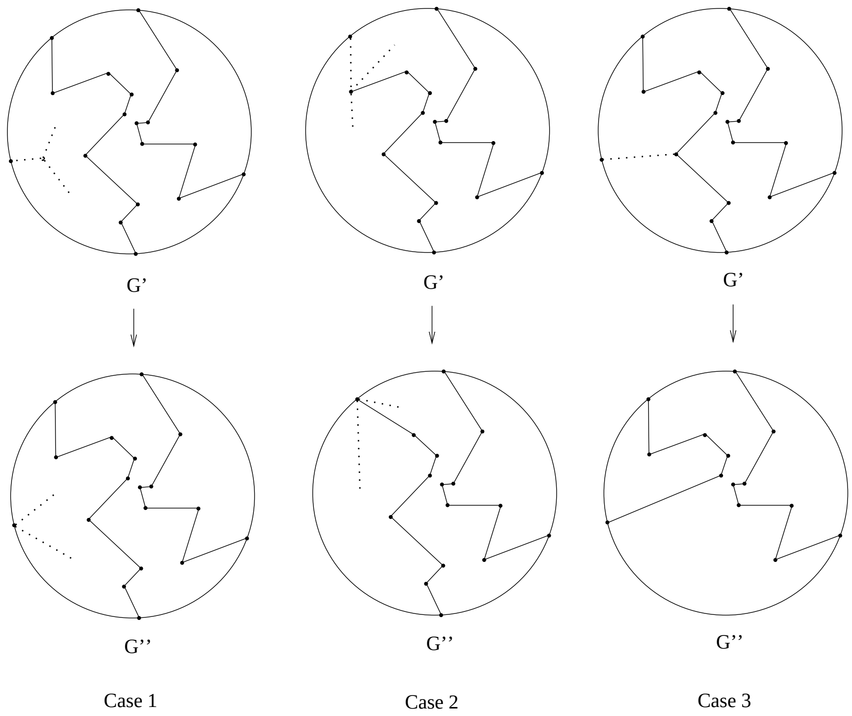

Figure 5: Maintaining two disjoint paths under spike promotion

The graph can be reduced in this manner until e_1 and e_2 are boundary-boundary edges, and since an iff relationship holds at each step it follows that the theorem holds for the original graph G.

Corollary 1 *An interior vertex in a critical circular planar network has at least three disjoint path to the boundary.*

Proof If all vertices in the neighborhood of the vertex are in the boundary then the theorem holds since there must be at least three vertices in the neighborhood. Assume one of the edges connected to the vertex does not go to the boundary. This edge, call it e, must have a simple boundary-boundary path through it (which can be proved by simply deleting some of the text in the proof of Theorem 2). Consider the two edges in this path on each side of e. By Theorem 2 these edges must have disjoint boundary-boundary paths. Now by taking the union of one of these paths, part of the other path, and e, the desired three disjoint boundary-boundary paths are constructed (see Figure 6).

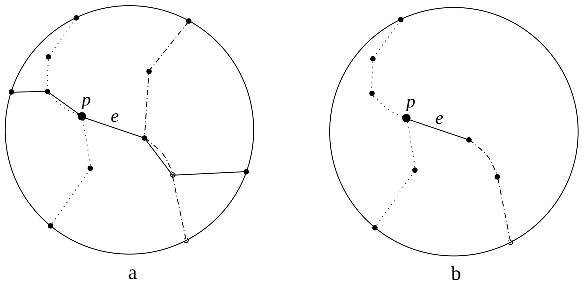

Figure 6: (a) P, e, and two disjoint boundary-boundary paths and (b) three disjoint boundary paths

References

[1] E. B. Curtis, D. Ingerman and J. A. Morrow, *Circular Planar Graphs and Resistor Networks*