Planarity of Networks with Four or Five Boundary Nodes

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1 Introduction

A *graph with boundary* is a triple $\Gamma = (\mathrm{V}, \mathrm{E}, \partial\mathrm{V})$, where Γ is a finite graph with V = the set of nodes, E = the set of edges where the conductivity γ acts, and ∂V = the non-empty subset of V called the boundary nodes where the current I is induced.

A *circular planar graph* is a graph with a boundary which is embedded in a disc in the plane so that the boundary nodes lie on the circle which bounds the disc, and the rest of Γ is in the interior of the disc. The boundary nodes will be labeled in circular order around the boundary of the disc.

A pair of sequences of boundary nodes $(P;Q) = (p_1, ..., p_k; q_1, ..., q_k)$ of a circular planar graph Γ such that the entire sequence $(p_1, ..., p_k, q_1, ...q_k)$ is in circular order is called a *circular pair*.

A circular pair $(P;Q)$ of boundary nodes is said to be *connected through* a circular planar graph Γ if there are k disjoint paths $\alpha_1, ..., \alpha_k$ in Γ such that α_i starts at p_i, ends at $q_{(k-i)+1}$, and passes through no other boundary nodes. We say that α is a *connection* from P to Q.

A *Y-Δ equivalence* is a geometric transformation that replaces three edges of a graph which form a **Y** connection with three edges forming a Δ connection. Electrical equivalence of the graph is maintained.

A *conductivity* on any graph Γ is a function γ which assigns to each edge e in E a positive real number $\gamma(e)$. A *resistor network* (Γ, γ) consists of a graph with boundary together with a conductivity function γ.

The *Kirchhoff matrix* $K = K(\Gamma, \gamma)$ of a network (Γ, γ) with n nodes numbered $v_1, ..., v_n$ is the $n \times n$ matrix constructed as follows.

1. If $i \neq j$ then $K_{i,j} = -\Sigma\gamma(e)$, where the sum is taken over all edges joining v_i to v_j. If there is no edge joining v_i to v_j then $K_{i,j} = 0$.

2. $K_{i,i} = \Sigma\gamma(e)$ where the sum is taken over all edges e with one endpoint at v_i and the other endpoint not v_i.

Thus, all diagonal entries of K are non-negative, and all off-diagonal entries are non-positive, and all row or column sums are 0.

The Λ Matrix For each voltage potential f defined at the boundary nodes, there is a unique extention of f, to all the nodes of Γ which satisfies Kirchhoff's current law at each interior node of a resistor network. We will call this unique extention u. In other words, if p is an interior node of (Γ, γ), then

$$\Sigma\gamma(e)(u(p) - u(q)) = 0$$

where the sum is taken over all edges e with one endpoint at p and the other endpoint $q \neq p$. This function f then gives a current $I = \{I(p) | p \in \partial\mathrm{V}\}$ into the network a the boundary nodes. The linear map which sends f to I is called the Dirichlet-to-Neumann map and is represented by an $n \times n$ matrix denoted Λ. Λ is called the *network response*.

All Λ matrices have the same properties as Kirchhoff matrices.

Diagonal entries are non-negative.

Off-diagonal entries are non-postive.

Row sums and column sums are 0.

THEOREM. *In addition, if Λ is the network response for a circular planar network, and $(P;Q) = (p_1, ..., p_k; q_1, ..., q_k)$ is a circular pair of boundary nodes, then*

(a) If $(P;Q)$ are not connected through Γ, then

$$det\Lambda(P;Q) = 0$$

.

(b) If $(P;Q)$ are connected through Γ, then

$$(-1)^{\frac{k(k+1)}{2}} det\Lambda(P;Q) > 0$$

.

$\Lambda(P;Q)$ *here refers to the matrix formed by taking only the rows $p_1, ..., p_k$ and columns $q_1, ..., q_k$ of Λ.*

Order is important. For more information on the Λ matrix of a circular planar graph, or any of the definitions above, please see [1].

The Medial Graph for a Circular Planar Network Suppose Γ = (V,E,∂V) is a circular planar graph with n boundary nodes $v_1, ..., v_n$. Γ is assumed to be embedded in the plane so that the boundary nodes occur in clockwise order around the circle C. To construct the medial graph $\mathcal{M}(\Gamma)$ first let m_e be the midpoint of each edge e in E. Next, place $2n$ points $t_1, t_2, ..., t_{2n}$ on C so that

$$t_1 < v_1 < t_2 < t_3 < v_2 < ... < t_{2n-1} < v_n < t_{2n} < t_1$$

in the clockwise circular order around C.

The vertices of $\mathcal{M}(\Gamma)$ consist of the points m_e for $e \in$E, and the points t_i for $i = 1, 2, ..., 2n$. Two vertices m_e and m_f are joined by an edge whenever e and f have a common vertex and e and f are incident to the same face in Γ. There is also one edge for each point t_j as follows. The point t_{2i} is joined bly an edge to m_e where e is the edge of the form $e = v_i r$ which comes first after arc$v_i t_{2i}$ in clockwise order around v_i. The point t_{2i-1} is joined by an edge to m_f where f is the edge of the form $f = v_i s$ which comes first after arc $v_i T_{2i-1}$ in counter-clockwise order around v_i.

The vertices of $\mathcal{M}(\Gamma)$ of the form m_e of are 4-valent; the vertices of the form t_i are 1-valent. An edge uv of $\mathcal{M}(\Gamma)$ has a *direct extension* vw if the edges uv and vw separate the other two edges incident on the vertex v. A path $u_0 u_1 ... u_k$ in $\mathcal{M}(\Gamma)$ is called a *geodesic arc* if each edge $u_{i-1}u_i$ has edge $u_i u_{i+1}$ as a direct extention. A geodesic arc is called a *geodesic* if either

(1) u_0 and u_k are points on the circle C; or

(2) $u_k = u_1$ and $u_{k-1}u_k$ has $u_0 u_1$ as a direct extention.

Two geodesics form a *lens* if they intersect each other more than once. If each geodesic in $\mathcal{M}(\Gamma)$ begins and ends on C and has no self-intersection, and if $\mathcal{M}(\Gamma)$ has no lenses, we will say that $\mathcal{M}(\Gamma)$ is *lensless*.

Theorem. Γ *is a critical circular planar graph, if and only if* $\mathcal{M}(\Gamma)$ *is lensless.*

Refer to [1].

The Key Lemma tells us how to find which endpoints belong to which geodesics by telling us the number of re-entrant geodesics between any two points a and b on the circle C. Points a and b divide C into two parts, D and F.

$$\mathrm{R}(D)=\mathrm{card}(D)\text{-}\mathrm{Black}(D)\text{-}\mathrm{Max}(D,F)$$

where $\mathrm{R}(D)$is the number of re-entrant geodesics in D, $\mathrm{card}(D)$ is the number of boundary nodes in D, $\mathrm{Black}(D)$ is the number of black intervals in D, and Max (D, F) is the size of the biggest disjoint connection between D and F (which can be found from Λ). Since a critical graph has a lensless medial graph, when we know the endpoints of the geodesics in the medial graph, we can find the medial graph, and from the medial graph we can find a graph which is Y-Δ equivalent to the original graph. For more information on recovering the medial graph, refer to [2].

2 Characterizing Networks that cannot be made Circular Planar

Definition. Two networks are called $\Lambda -$ *equivalent* if they have the same Λ matrix. This is sometimes referred to as being electrically equivalent.

2.1 The Four Node Case

Theorem. *A* 4×4 *Kirchhoff matrix is either the* Λ *matrix for a circular planar network, or it can be made so by one row switch and the identical column switch.*

In other words, a network with 4 boundary nodes is Λ -equivalent to a circular planar network, or its boundary nodes can be renumbered so that it is.

Proof. Consider the Kirchhoff matrix

$$K = \begin{pmatrix} a+b+c & -a & -b & -c \\ -a & a+d+e & -d & -e \\ -b & -d & b+d+f & -f \\ -c & -e & -f & c+e+f \end{pmatrix}$$

Let $K^{2,3}$ denote K with rows 2 and 3 switched, and columns 2 and 3 switched.

$$K^{2,3} = \begin{pmatrix} a+b+c & -b & -a & -c \\ -b & b+d+f & -d & -f \\ -a & -d & a+d+e & -e \\ -c & -f & -e & c+e+f \end{pmatrix}$$

Let $K^{3,4}$ denote K with rows 3 and 4 switched and columns 3 and 4 switched.

$$K^{3,4} = \begin{pmatrix} a+b+c & -a & -c & -b \\ -a & a+d+e & -e & -d \\ -c & -e & c+e+f & -f \\ -b & -d & -f & b+d+f \end{pmatrix}$$

It is easy to check that $K^{2,3}$ and $K^{3,4}$ are also Kirchhoff matrices.

To prove that a Kirchhoff matrix is a Λ matrix of a circular planar network, it suffices to show that the determinantal properties hold. Thus,

K is a Λ matrix $\Leftrightarrow be - cd \leq 0$ and $be - af \leq 0$

$K^{2,3}$ is a Λ matrix $\Leftrightarrow af - cd \leq 0$ and $af - be \leq 0$

$K^{3,4}$ is a Λ matrix $\Leftrightarrow cd - be \leq 0$ and $cd - af \leq 0$

When $be = 0$, K is a Λ matrix. When $af = 0$, $K^{2,3}$ is a Λ matrix. When $cd = 0$, $K^{3,4}$ is a Λ matrix.

Table 1 examines all possible combinations of determinants $(be - cd)$ and $(be - af)$ for the original K, when $be > 0$, $cd > 0$ and $af > 0$.

Note that

$$af - cd = (be - cd) - (be - af) \qquad af - be = -(be - af)$$

$$cd - af = (be - af) - (be - cd) \qquad cd - be = -(be - cd)$$

Thus, Table 1 demonstrates that for all possible values of the Kirkhoff matrix K, a Λ matrix can be made by at most one row-row and column-column switch. *QED*.

$be-cd$		$be-af$	$af-cd$	$af-be$	$cd-be$	$cd-af$	**Λ matrix**
$+$	$<$	$+$	$-$	$-$	$-$	$+$	$K^{2,3}$
$+$	$=$	$+$	0	$-$	$-$	0	$K^{2,3}, K^{3,4}$
$+$	$>$	$+$	$+$	$+$	$-$	$-$	$K^{3,4}$
$+$	$>$	0	$+$	0	$-$	$-$	$K^{3,4}$
$+$	$>$	$-$	$+$	$+$	$-$	$-$	$K^{3,4}$
0	$<$	$+$	$-$	$-$	0	$+$	$K^{2,3}$
0		0					K
0		$-$					K
$-$	$<$	$+$	$-$	$-$	$+$	$+$	$K^{2,3}$
$-$		0					K
$-$		$-$					K

Table 1: All Possible Combinations of Off-Diagonal Determinants in the 4 Boundary Node Case

2.2 The Five Node Case

Given the general Kirchhoff matrix,

$$K = \begin{pmatrix} \sum & -a & -b & -c & -d \\ -a & \sum & -e & -f & -g \\ -b & -e & \sum & -h & -j \\ -c & -f & -h & \sum & -k \\ -d & -g & -j & -k & \sum \end{pmatrix}$$

for the ordering **12345** of a network with five boundary nodes, Table 2 displays all the possible circular planar orderings. Listed with them are the determinants that must be ≤ 0 in order for the reordered network to be Λ-equivalent to a circular planar network. The different combinations of determinant pairs are labelled 0, 1 or 2 in the shorthand column. For example, for det 1

0 refers to the pair (fj-gh) and (fj-ek),

1 refers to the pair (gh-fj) and (gh-ek),

2 refers to the pair (ek-fj) and (ek-gh).

embedding sequence	*det* 1	*det* 2	*det* 3	*det* 4	*det* 5	**shorthand det.sequence**
12345	$fj - gh$ $fj - ek$	$cj - bk$ $cj - dh$	$cg - df$ $cg - ak$	$bg - aj$ $bg - de$	$bf - ce$ $bf - ah$	00000
12354	$gh - fj$ $gh - ek$	$dh - cj$ $dh - bk$	$df - cg$ $df - ak$	$bg - aj$ $bg - de$	$bf - ce$ $bf - ah$	12100
12435	$ek - fj$ $ek - gh$	$bk - cj$ $bk - dh$	$cg - df$ $cg - ak$	$bg - aj$ $bg - de$	$ce - bf$ $ce - ah$	21001
12453	$gh - fj$ $gh - ek$	$dh - cj$ $dh - bk$	$cg - df$ $cg - ak$	$de - bg$ $de - aj$	$ce - bf$ $ce - ah$	12021
12534	$ek - fj$ $ek - gh$	$bk - cj$ $bk - dh$	$df - cg$ $df - ak$	$de - bg$ $de - aj$	$bf - ce$ $bf - ah$	21120
12543	$fj - gh$ $fj - ek$	$cj - bk$ $cj - dh$	$df - cg$ $df - ak$	$de - bg$ $de - aj$	$ce - bf$ $ce - ah$	00121
13245	$gh - fj$ $gh - ek$	$cj - bk$ $cj - dh$	$cg - df$ $cg - ak$	$aj - bg$ $aj - de$	$ah - bf$ $ah - ce$	10012
13254	$fj - gh$ $fj - ek$	$dh - cj$ $dh - bk$	$df - cg$ $df - ak$	$aj - bg$ $aj - de$	$ah - bf$ $ah - ce$	02112
13425	$ek - fj$ $ek - gh$	$cj - bk$ $cj - dh$	$ak - cg$ $ak - df$	$aj - bg$ $aj - de$	$ce - bf$ $ce - ah$	20211
13524	$ek - fj$ $ek - gh$	$dh - cj$ $dh - bk$	$ak - cg$ $ak - df$	$de - bg$ $de - aj$	$ah - bf$ $ah - ce$	22222
14235	$gh - fj$ $gh - ek$	$bk - cj$ $bk - dh$	$ak - cg$ $ak - df$	$bg - aj$ $bg - de$	$ah - bf$ $ah - ce$	11202
14325	$fj - gh$ $fj - ek$	$bk - cj$ $bk - dh$	$ak - cg$ $ak - df$	$aj - bg$ $aj - de$	$bf - ce$ $bf - ah$	01210

Table 2: Determinant Combinations for all possible Re-embeddings
Note that this particular labelling of a determinant pair refers to the original position of the excluded node, not its new location in the reordering.

LEMMA. *At least three determinants must change in sign when the boundary nodes of a network with five boundary nodes are reordered. Moreover, determinants must change in specific ways in relation to each other.*

Proof. Follows directly from Table 2. See shorthand column.

THEOREM. *There exist some* 5×5 *Kirchhoff matrices whose rows and columns cannot be reordered so that they are the* Λ *matrices of circular planar networks.*

Proof. The total number of unique circular orderings of 5 points is

$$\frac{5!}{5 \times 2} = 12$$

with 5! being the number of possible linear arrangements of five nodes, divided by 5 because the starting node is irrelevent and divided by two because the direction of the ordering is also irrelevent. Now, each pair of determinants has three possible combinations, only one of which is guarenteed to have both determinants ≤ 0 at any one time (see proof of previous Theorem). Therefore, the total number of determinant combinations is 3^d where d is the number of determinant pairs, 5 in this case. Not all of these $3^5 = 243$ combinations are algebraically possible, as will be demonstrated further on. However, since $243 \gg 12$ orderings, some algebraically possible combinations of determinants will not have an ordering which produces them.

Consider the counterexample:

$$K = \begin{pmatrix} 105 & -2 & -100 & -1 & -2 \\ -2 & 6 & -2 & -1 & -1 \\ -100 & -2 & 105 & -2 & -1 \\ -1 & -1 & -2 & 5 & -1 \\ -2 & -1 & -1 & -1 & 5 \end{pmatrix}$$

Three pairs of determinants are > 0, and therefore should not be changed, but both values of each of the other two pairs are > 0. Since all reorderings change the signs of at least three determinant pairs, no ordering will make K into a Λ for a circular planar graph. *QED*.

Table 3 shows all the possible combinations of determinantal values from the general 5×5 Kirchhoff matrix that **can** be re-ordered so as to be the Λ matrix of a circular planar network, as well as the corresponding re-orderings.

To help understand the symbols used, the second row of the table says that

$(fj - gh) \leq 0$ and $(fj - ek) \leq 0$

$(cg - df) \geq (cg - ak)$ and $(cg - df) \geq 0$

$(bg - aj) \leq (bg - de)$ and $(bg - de) \geq 0$

Any network whose Λ matrix determinants do not fit into this table is necessarily noncircular or nonplanar.

3 Five Nodes on the Annulus

Consider a 5×5 Λ matrix whose determinants do not match any of the combinations in Table 3. The network such a Λ represents cannot be reordered so as to be Λ equivalent to a circular planar graph. Therefore, either the circular nature of the boundary, or the planarity of the network must change. Let us examine networks which can be embedded in an annular planar region.

Definition. An *annular planar graph* is a graph with boundary which is embedded in an annulus so that the boundary nodes lie on either of the two circles which bound the annulus.

A list of boundary nodes $v_1, ..., v_k$ are said to be in *numerical circular order* if $v_1 ... v_k$ embedded in a circular boundary would be in circular order. Nodes on either circle bounding the annulus should be in numerical circular order.

Definition. For a five node annular planar network, two pairs of boundary nodes (p_1, p_2) and (q_1, q_2) are called *connected* if there exist disjoint paths from $p_1$to q_2 and from p_2 to $q_1$when $(p_1, p_2)(q_1, q_2)$ are in numerical circular order.

It should be noted that this connectivity exists when

$$(-1)^{\frac{(2)(3)}{2}} det\Lambda((p_1, p_2), (q_1, q_2)) > 0$$

Two pairs of nodes are called *cross-connected* if there exist disjoint paths from p_1 to q_1 and from p_2 to q_2 when $(p_1, p_2)(q_1, q_2)$ are in numerical circular order.

$fj - gh$ $fj - ek$	$cj - bk$ $cj - dh$	$cg - df$ $cg - ak$	$bg - aj$ $bg - de$	$bf - ce$ $bf - ah$	**switch needed** **to make a planar** Λ
$-$ $-$	$-$ $-$	$-$ $-$	$-$ $-$	$-$ $-$	**none**
$-$ $-$	$-$ $-$	$++$ $+/-$	$+/-$ $++$	$++$ $+/-$	12543
$-$ $-$	$+/-$ $++$	$++$ $+/-$	$++$ $+/-$	$+/-$ $++$	13254
$-$ $-$	$++$ $+/-$	$+/-$ $++$	$++$ $+/-$	$-$ $-$	14325
$+/-$ $++$	$-$ $-$	$+/-$ $++$	$++$ $+/-$	$++$ $+/-$	13425
$+/-$ $++$	$+/-$ $++$	$+/-$ $++$	$+/-$ $++$	$+/-$ $++$	13524
$+/-$ $++$	$++$ $+/-$	$-$ $-$	$-$ $-$	$++$ $+/-$	12435
$+/-$ $++$	$++$ $+/-$	$++$ $+/-$	$+/-$ $++$	$-$ $-$	12534
$++$ $+/-$	$-$ $-$	$-$ $-$	$++$ $+/-$	$+/-$ $++$	13245
$++$ $+/-$	$+/-$ $++$	$-$ $-$	$+/-$ $++$	$++$ $+/-$	12453
$++$ $+/-$	$+/-$ $++$	$++$ $+/-$	$-$ $-$	$-$ $-$	12354
$++$ $+/-$	$++$ $+/-$	$+/-$ $++$	$-$ $-$	$+/-$ $++$	14235

Table 3: Determinantal Relations that Suggest Re-embedding

This connectivity exists when

$$(-1)^{\frac{(2)(3)}{2}} det\Lambda((p_1, p_2), (q_1, q_2)) < 0$$

because the wrong sign in the determinant indicates that a disjoint connection exists and is numerically dominant between pairs not in circular order. Nothing is indicated about the existence of connections in numerical circular order, except that if they exist, they are weaker than the cross-connection.

DEFINITION. A graph is said to be *critical* if contracting or breaking any edge breaks a connection or a cross-connection indicated by Λ.

Note that Y-Δ transformations can only be performed on portions of the graph that are isomorphic to a portion of a circular planar graph.

For clarity, let us consider a determinant pair to be *problematic* if at least one of the two determinants is positive.

3.1 Two Pairs of Problematic Determinants: Adjacent Pairs

DEFINITION. Let the *index* of a pair of 2×2 determinants in a five boundary node network be equal to the position of the node excluded from the two determinants.

Two pairs of determinants, index i and j are called *adjacent* if i and j are next to each other in circular ordering.

Consider a graph, Γ with nodes **1**, **3**, **4** and **5** on the outer circle of an annulus, **2** on the innner circle, and one interior node **p** (See Figure 1).

Let its Kirchhoff matrix be

$$K = \begin{pmatrix} a+b+c+d & -a & -b & 0 & -c & -d \\ -a & a+e+f & -e & 0 & 0 & -f \\ -b & -e & b+e+g+h & -g & 0 & -h \\ 0 & 0 & -g & g+j & 0 & -j \\ -c & 0 & 0 & 0 & c+k & -k \\ -d & -f & -h & -j & -k & \Sigma_p \end{pmatrix}$$

The determinants are as follows:

1	2	3	4	5
$-fgk\frac{1}{\Sigma_p}$	$-bjk\frac{1}{\Sigma_p}$	$-cfj\frac{1}{\Sigma_p}$	$(bfk-ahk)\frac{1}{\Sigma_p}$	$(bfj-dej)\frac{1}{\Sigma_p}$
$-ejk\frac{1}{\Sigma_p}$	$-(chj+dgk)\frac{1}{\Sigma_p}-cg$	$-ajk\frac{1}{\Sigma_p}$	$(bfk-dek$ $-cfh)\frac{1}{\Sigma_p}-ce$	$(bfj-dfg$ $-ahj)\frac{1}{\Sigma_p}-ag$

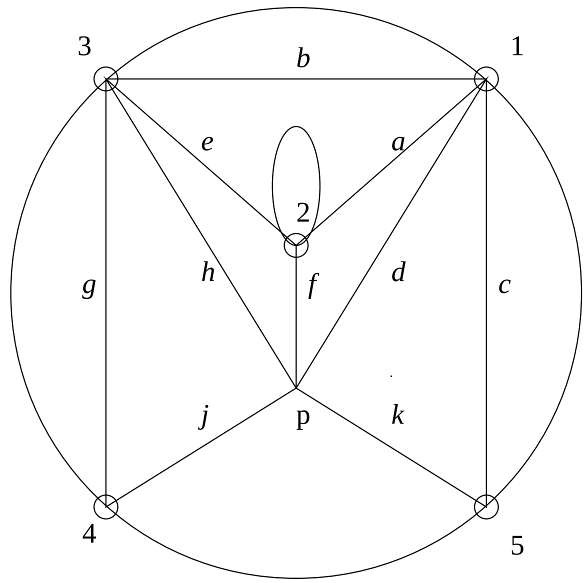

Figure 1: Example of an Annular Planar Network

Determinant pairs 1, 2 and 3 are necessarily negative, but pairs 4 and 5 can be either positive or negative. If we make determinant pairs 4 and 5 positive, the result is a network which cannot be re-embedded to be Λ-equivalent to a circular planar network. Moreover, the two pairs of problematic determinants are adjacent.

Determinant 4 indicates that the cross-connections linking **1** to **3** and **2** to **5** exist and are stronger than any possible connection linking **1** to **5** and **2** to **3**, or linking **1** to **2** and **3** to **5**. Determinant 5 likewise indicates that cross-connections linking **1** to **3** and **2** to **4** must exist. The simplest way to achieve this is to link nodes **1** and **3** directly and give $\gamma_{1,3}$ a large value.

Medial Graph for the Annular Planar Graph Consider the medial graph, $\mathcal{M}(\Gamma)$ as shown in Figure 2. Two geodesics form a lens, but it is a necessary lens because it contains the hole in the annulus. Two other geodesics cross from one boundary circle to another. In order to recover the medial graph from Λ, we need to make the annulus imitate the circular planar case. To this end, mark the ten geodesic endpoints, two around each of the boundary nodes. Now imagine cutting from the white half of the interior circle to the white interval between geodesics 2 and 5. This gives the graph

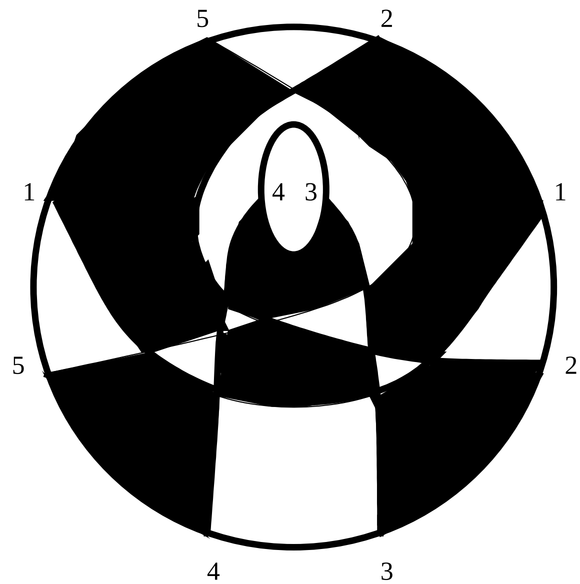

Figure 2: Medial Graph for Annular Planar Network from Fig. 1.

a new boundary which is topologically equivalent to a circular planar graph. This cut has certain effects upon the connectivity of the graph, which are important to note.

1. Nodes **1** and **3** are no longer directly connected. Therefore, since the graph we are recovering is critical, any circular planar connection of pairs which involve connecting **1** to **3** will now be broken.

2. Add two new nodes, **A** and **B**, and their four accompanying geodesic endpoints, to the boundary, putting **A** between nodes **1** and **2**, and **B** between nodes **2** and **3** along the edges formed by the cut (See Figure 3). These represent the two halves of the connection between nodes **1** and **3**, which means that **A** is connected only to **1** and likewise, **B** is connected only to **3**.

3. All four determinants which were previously positive, were dependent upon the $\partial - \partial$ connection between nodes **1** and **3**. The values of these determinants may now be zero, or they may be negative. Nothing should be assumed about these connections when they come up in the process of finding the z-sequence of the pseudograph.

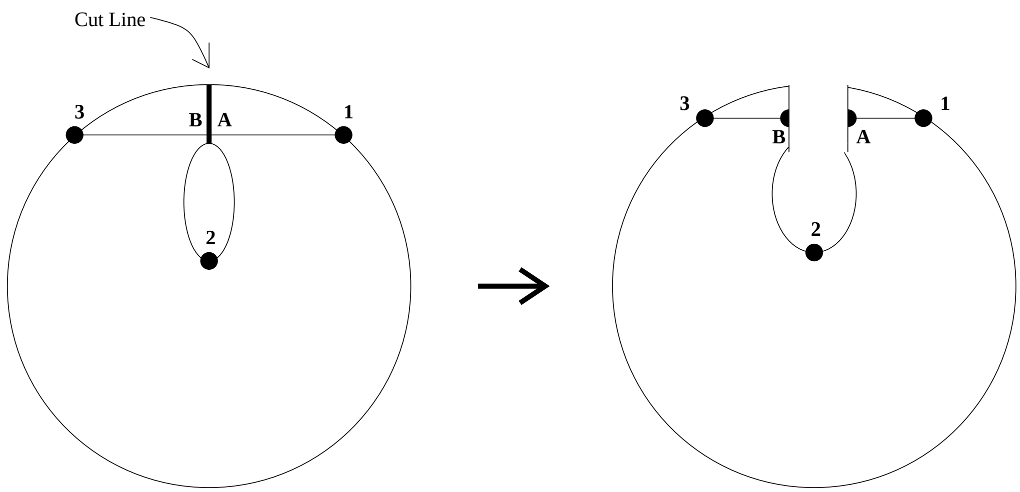

Figure 3: Creating the Boundary of the Pseudo-Medial Graph from the Original Boundary

With these important changes in connectivity kept firmly in mind, find the medial graph $\mathcal{M}(\Gamma')$ of the pseudograph $\Gamma' =$**(1 A 2 B 3 4 5)**.

Paste the segments around **A** and **B**, which were formed by the cut, back together. This will create a lens because both **A** and **B** were treated as boundary spikes. The lens can be easily eliminated by switching the two geodesic endpoints around **B**. When all geodesic endpoints are connected to their partners without crossing the line of the cut, the true medial graph for the Λ matrix has been realized.

Recoverability

Theorem. *If two adjacent pairs of determinants of the Λ matrix have incorrect signs, and the remaining three have the correct signs then there exists a network recoverable from Λ which is embedded on an annulus with one node on one circle and four nodes on the other circle.*

The proof lies in the actual method of recovery, which is laid out below.

Re-number the nodes so that the determinants with incorrect signs have index 4 and 5. Now we have a base structure for the network, with node **2** on the inner circle of the annulus and a $\partial - \partial$ connection between nodes **1** and **3**.

Find the medial graph, as described above. Draw the graph from the medial graph, and proceed to recover the conductivities using Λ. The graph will be recoverable, because the pseudo-medial graph does not have any lenses, so the circular planar pseudo-graph would be recoverable, and therefore the graph is recoverable.

Example of Recovery. To start with, we want some Λ matrix with two adjacent problematic determinant pairs. Let

$$\Lambda_0 = \begin{pmatrix} 5 & -2 & -1 & -1 & -1 \\ -2 & 12 & -2 & -1 & -7 \\ -1 & -2 & 8 & -4 & -1 \\ -1 & -1 & -4 & 7 & -1 \\ -1 & -7 & -1 & -1 & 10 \end{pmatrix}$$

The determinant pairs are:

1	**2**	**3**	**4**	**5**
−27	0	6	5	−1
−1	−3	5	5	−7

The first step is to relabel the nodes without changing circular ordering so that the problematic deterinants are index 4 and 5. Renumber node **1** as **2**, and **2** as **3** and so on. The new

$$\Lambda = \begin{pmatrix} 10 & -1 & -7 & -1 & -1 \\ -1 & 5 & -2 & -1 & -1 \\ -7 & -2 & 12 & -2 & -1 \\ -1 & -1 & -2 & 8 & -4 \\ -1 & -1 & -1 & -4 & 7 \end{pmatrix}$$

The determinants are now as desired:

1	**2**	**3**	**4**	**5**
−1	−27	0	6	5
−7	−1	−3	5	5

Now we can assume a base structure on the annulus with a strong connection between nodes **1** and **3**, and node **2** on the inner circle of the annulus

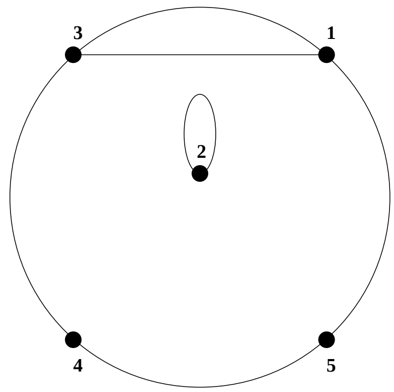

Figure 4: Base Structure for the Network

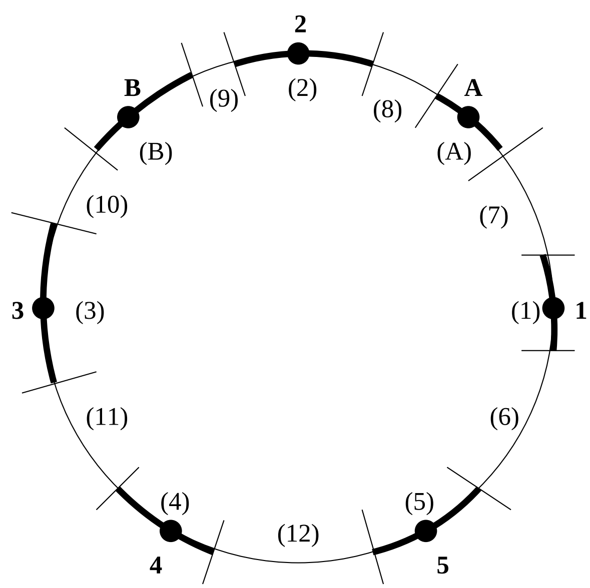

Figure 5: Boundary Used to Find the Pseudo-Medial Graph

(See Figure 4). Set up the pseudo-medial graph **1 A 2 B 3 4 5** as in Figure 5. The numbers in parentheses label the intervals on the boundary, which alternate between black and white.

Using the Key Lemma

R(D)=card(D)-Black(D)-Max(D, F)

we can find R(D), the number of reentrant geodesics in D. Note that Max(D, F) can be found using Λ as in the circular planar case, by examining the determinants as shown in the definitions at the beginning of Section 3. Recall that a cross-connection does not necessarily indicate a connection, and any connections involving a connection between nodes **1** and **3** are now broken. Also, when calculating the largest disjoint connection between D and F, it must be kept in mind that node **A** connects to node **1**, and node **B** connects to node **3**.

To keep track of where the Key Lemma is being applied, the notation will be (X-Y;Z) where X and Y are the two intervals chosen to split the graph into sides D and F, and Z is an interval on the side D under consideration.

(8-12;1)	$6-3-2=1$	
(A-12;1)	$5-3-2=0$	Label endpoint between intervals (8) and (A) as an endpoint of geodesic 1
(8-5;1)	$5-3-1=1$	
(8-6;1)	$4-2-1=1$	
(8-1;A)	$3-2-0=1$	The endpoint between intervals (1) and (6) is the other endpoint of geodesic 1.
(2-12;1)	$7-4-1=2$	Label endpoint between intervals (2) and (8) as an endpoint of geodesic 2.
(2-5;1)	$6-4-1=1$	The endpoint between intervals (5) and (12) is the other endpoint of geodesic 2, because the re-entrant geodesic left is geodesic 1. This means that we can label the endpoint between intervals (A) and (7) as an endpoint of geodesic 3, the endpoint between intervals (1) and (6) as an endpoint of geodesic 4, and the endpoint between intervals (6) and (5) as an endpoint of geodesic 5.
(A-4;1)	$6-4-?=?$	A connection may or may not exist between (**2 3**) (**5 1**). All that Λ reveals is the cross-connection (**2 3**) (**1 5**).

(A-11;1)	$7-4-2=1$	A geodesic re-entered either between interals (12) and (4) or between interals (4) and (11).
(7-11;1)	$6-3-3=0$	Label the endpoint between intervals (A) and (7) as an endpoint of geodesic 3. This is the geodesic which re-entered either between intervals (12) and (4) or between intervals (4) and (11). Note that the endpoint which does not belong to geodesic 3 is not re-entrant. Let us call it an endpoint of geodesic 6.
(1-3;5)	$6-4-1=1$	
(1-11;5)	$5-3-2=0$	
(3-6;5)	$5-3-1=1$	
(3-5;12)	$4-3-1=0$	This means that the endpoint between intervals (3) and (11) belongs to geodesic 5.
(1-10;12)	$7-4-2=1$	Note the connection between nodes **3** and **B**. The re-entrant geodesic is geodesic 5, so we can say that the endpoint between intervals (3) and (10) belongs to a new geodesic, 7.
(10-6;12)	$6-3-2=1$	
(10-5;12)	$5-3-2=0$	As expected, geodesic 5 is no longer re-entrant.
(B-5;12)	$6-4-?=?$	This is the other place where connectivity is unknown, because any connection that exists between (**3 4**) and (**1 2**) was masked by the cross-connection.
(B-11;3)	$3-2-1=0$	This proves that the endpoint between intervals (B) and (10) does not belong to geodesic 7. It must belong to 4 or 6.
(9-11;3)	$4-2-1=1$	
(3-9;B)	$3-2-0=1$	
(10-9;B)	$2-1-1=0$	The endpoint between intervals (B) and (9) belongs to geodesic 7.
(12-B;3)	$5-3-2=0$	This proves that the endpoint between intervals (B) and (10) does not belong to geodesic 6. It must belong to geodesic 4.

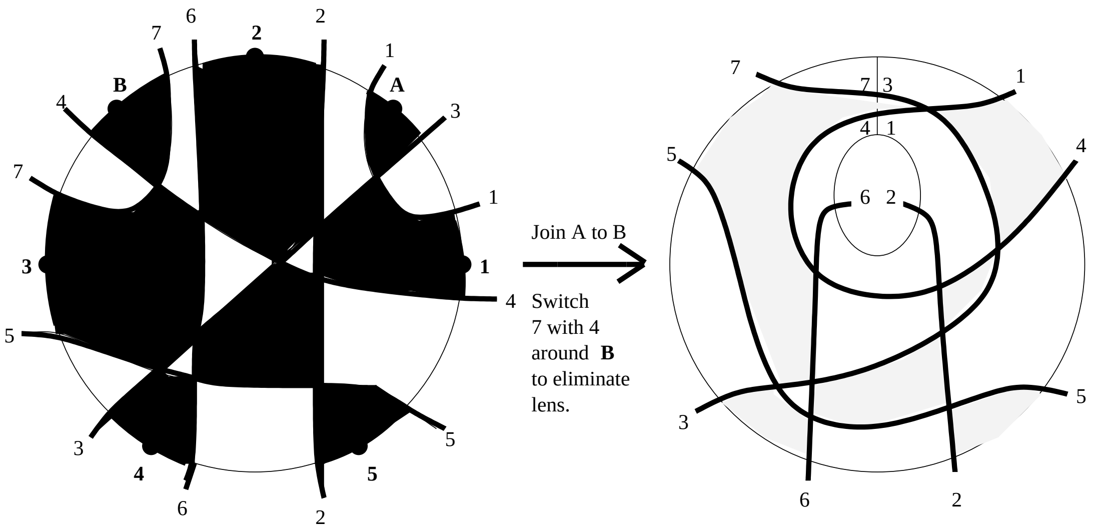

Figure 6: Recovered Pseudo-Medial Graph and Actual Medial Graph

(9-6;A)	$6 - 3 - ? = ?$	Note that despite the inconclusive answer here, we know that geodesics 1, 2, 4, 5, and 7 have been found, and geodesic 3 is pinned down to one of two endpoints, while the other endpoint is known, so the endpoint between intervals (9) and (2) must belong to geodesic 6.
(2-4;3)	$6 - 4 - 1 = 1$	This re-entrant geodesic is geodesic 7, so geodesic 6 does not re-enter. Therefore, the endpoint between intervals (4) and (12) belongs to geodesic 6, which leaves the endpoint between intervals (4) and (11) for geodesic 3.

The pseudo-medial graph having been recovered, as shown in Figure 6, we can paste the original cut back together and recover the graph (Figure 7). Using Λ, we can recover the conductivities.

3.2 Two Pairs of Problematic Determinants: General Case

THEOREM. *Given a* 5×5 Λ *matrix, if two pairs of determinants are problematic, then they must be adjacent.*

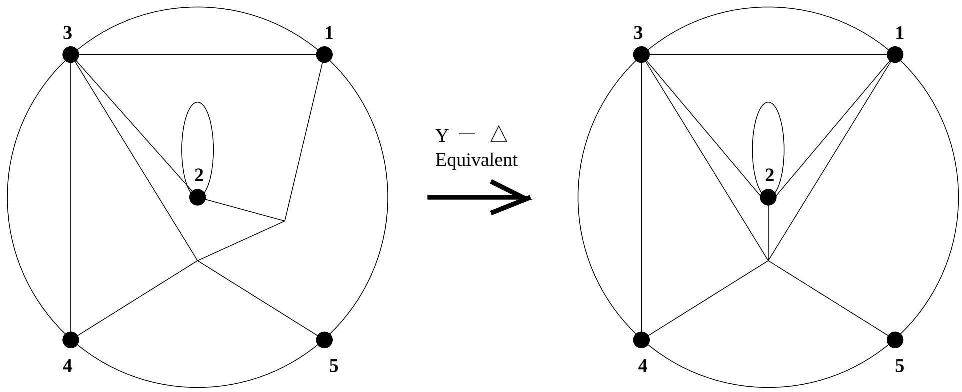

Figure 7: Actual Graph for the Annular Network with Λ Response Matrix

Proof. In any combination of 5 determinant pairs with exactly two problematic pairs, they must either be adjacent or separated by one non-positive determinant pair. In the latter case, the network can be renumbered so that the problematic pairs of determinants have index 3 and 5.

Now, referring to the general Kirchhoff matrix in Section 2.2, this means that at least one of the following statements is true:

Case 1: $cg - df > 0$ and $bf - ce > 0$

Case 2: $cg - df > 0$ and $bf - ah > 0$

Case 3: $cg - ak > 0$ and $bf - ce > 0$

Case 4: $cg - ak > 0$ and $bf - ah > 0$

All determinants of index 1, 2 and 4 are nonpositive, by assumption. Recall that $a, b, c, d, e, f, g, h, j, k$ are all nonnegative, by definition of Λ. We will proceed by cases to show that a determinant assumed to be nonpositive is necessarily positive.

Case 1: $cg - df > 0$ and $bf - ce > 0$ Now, since $cg - df > 0$, $g \neq 0$ so $c > \frac{df}{g}$ Substituting this into $bf - ce > 0$ gives

$$bf - \left(\frac{de}{g}\right) f > 0$$

We know that since $bf - ce > 0$, $f \neq 0$ so we can divide both sides by f:

$$bg - de > 0$$

But bg-de is a determinant of index 4, so this is a contradiction.

Case 2: $cg - df > 0$ and $bf - ah > 0$ Since its index is 4, $bg - aj \leq 0$ so $b \leq \frac{aj}{g}$ Substituting this into $bf - ah > 0$ gives

$$a\left(\frac{jf}{g}\right) - ah > 0$$

Now, since $g \neq 0$ and $b \neq 0$ in order to make the assumed equations positive, then $bg > 0$ so $a \neq 0$ if $bg - aj \leq 0$ Therefore, we can divide both sides of our inequality by a:

$$fj - gh > 0$$

But $fj - gh$ is a determinant of index 1, so this is a contradiction.

Case 3: $cg - ak > 0$ and $bf - ce > 0$ Since its index is 4, $bg - aj \leq 0$ so $g \leq \frac{aj}{b}$ Substituting this into $cg - ak > 0$ gives

$$a\left(\frac{cj}{b}\right) - ak > 0$$

As before, we know that $a \neq 0$ so we can divide both sides by a:

$$cj - bk > 0$$

But $cj - bk$ is a determinant of index 2, so this is a contradiction.

Case 4: $cg - ak > 0$ and $bf - ah > 0$ Since its index is 1, $fj - gh \leq 0$ Now, since $g \neq 0$ and $b \neq 0$ then $bg > 0$. But $bg - aj \leq 0$ because it is a determinant with index 4. So $j \neq 0$. Therefore, we can divide both sides of the inequalty $fj - gh \leq 0$ by j, getting $f \leq \frac{gh}{j}$. Substituting this into $bf - ah > 0$ gives

$$\left(\frac{bg}{j}\right)h - ah > 0$$

Now we know that $j \neq 0$, and $f \neq 0$ since $bf - ah > 0$, so $fj > 0$. However, $fj - gh \leq 0$ so $h \neq 0$ as well. This means we can divide both sides of our inequality by h:

$$bg - aj > 0$$

But $bg - aj$ is a determinant of index 2, so this is a contradiction.

Therefore, since all cases possible where two problematic pairs of determinants are separated by a 'good' determinant lead to contradictions, two pairs of problematic determinants must be adjacent. *QED*.

3.3 Sweeping Generality for the Five Node Case

THEOREM. *A 5×5 Kirchhoff matrix which is not the Λ matrix for a circular planar network, and cannot be made so by reordering the boundary nodes, is the Λ matrix for a five node network on the annulus.*

Proof. As stated above, there are $3^5 = 243$ possible combinations of five determinant pairs with

- $\ominus$ Both determinants in the pair less than or equal to 0; not problematic, *or*
- $+$ The first determinant greater than 0 and greater than the second determinant, *or*
- $\oplus$ The second determinant greater than 0 and greater than or equal to the first determinant.

However, not all 243 of these combinations are algebraically possible. Note that in order to avoid testing the same determinant combinations twice for impossibility, we assume strict inequality for the $+$ relationship. For actual re-embeddings, two equal determinants can be treated as being either $+$ or $\oplus$. It has already been shown that the combination

$$\ominus + \ominus + \ominus$$

is not algebraically possible, and causes a contradiction between the terms of the determinants. Therefore, in order to examine each of the 243 possible determinantal combinations and determine which actually could belong to a Kirchhoff matrix, follow these steps:

1. If the combination of determinants in question belongs to the list of combinations which indicate that reordering would make the network Λ-equivalent to a circular planar network, then note this and move along. These combinations are not only possible, but they work out very pleasantly.

2. If the combination of determinants in question does not belong to the reordering list, then examine how all 12 reorderings affect the combination. If any reordering results in only two, non-adjacent, problematic determinant pairs, then the original combination of determinant pairs is not algebraically possible. This can be shown by algebraic manipulations of the inequality statements.

   For example,

   $$+ + + + \oplus$$

   reordered **13254** has determinants

   $$+ \ominus \oplus \ominus \ominus$$

   which indicate an algebraic impossibility. The original determinant combination, applied to the general 5×5 Kirchhoff matrix in Section 2.2, gives the following statements:

   $$\begin{array}{c|c|c|c|c} fj - gh > 0 & cj - bk > 0 & cg - df > 0 & bg - aj > 0 & ce \geq ah \\ ek > gh & dh > bk & ak > df & de > aj & bf - ah > 0 \end{array}$$

   This tells us that f, j, c, g, and b are all non-zero, and consequently e, k, d, h, and a are also non-zero. So $b < \frac{dh}{k}$ since $bk < dh$. Therefore,

   $$bf - ah < \frac{dh}{k}f - ah = \frac{h}{k}(df - ak)$$

   Since $df < ak$, we know $bf - ah < 0$. But $bf - ah > 0$ so this combination is impossible.

   In the same way, any other combination which gives two nonadjacent problematic determinants when reordered can be shown to be algebraically impossible.

3. The only two cases in which no reordering of the determinant combination gives fewer than 3 problematic determinants are the combinations

$$+ + + + +$$

and

$$\ominus + \ominus \oplus \oplus$$

which can be shown to be impossible by algebraic manipulations of the determinant inequalities. The proofs of these algebraic impossibilities are left to the reader, and the appendix.

4. If reorderings of the determinant combination do not produce combinations containing only two problematic determinants, which are nonadjacent, but do produce combinations containing only one problematic determinant or two problematic determinants which are adjacent, then one of these reorderings can be embedded in the annulus, following the recovery directions above.

Therefore, all the possible combinations of determinants in a 5×5 Kirchhoff matrix belong to the Λ matrix of a network which can be embedded either as a circular planar network or as an annuluar planar network. For reference, all necessary calculations are presented in Appendix A. QED.

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[1] E. B. Curtis, D. Ingerman and J. A. Morrow. *Circular Planar Graphs and Resistor Networks.*

[2] Chris Stakewicz. *Recovering the Geometry of Circular Planar Resistor Networks from Boundary Measurements.*