RECOVERING THE GEOMETRY OF CIRCULAR PLANAR RESISTOR NETWORKS FROM BOUNDARY MEASUREMENTS

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Abstract. The following is a step by step algorhithm used to recover the "unknown" geometry of a circular planar graph. The bulk of this project is devoted towards **Theorem 4.1.10**, which relates connections between two boundary nodes in circular planar graph to the rank of a submatrix in the Dirichlet-to-Neumann map denoted by Λ. Briefly, the Dirichlet-to-Neumann map is a function which relates boundary information to the interior of a circular planar resistor network. More information regarding the Dirichlet-to-Neumann map can be found in [1]. A computer program writen in *Mathematica* 3.0 accompanies this presentation in section 6, as well as a complete *Mathematica* "package" format including examples of Dirichlet-to-Neumann maps with graphic displays of the resulting circular planar graph at the end of this paper. **1. Introduction.** A graph with a boundary is a triple Γ = (V, E, ∂V), where Γ is a finite graph with V = the set of nodes, E = the set of edges where the conductivity γ acts, and ∂V = the non-empty subset of V called the boundary nodes where the current I is induced. Γ is allowed to have multiple edges (i.e., more than one edge between two nodes) or loops (i.e., an edge joining a node to itself). Within the content of this paper, we will not be looking at loops, since in previous articles, it was noted that loops can be eliminated to produce electrically equivalent graphs.

A circular planar graph is a graph with a boundary which is embedded in a disc **D**in the plane so that the boundary nodes lie on the circle C which bounds **D**, and the rest of Γ is in the interior of **D**. The boundary nodes will be labelled v_1 ,..., v_n in the (clockwise) circular order around C. A pair of sequences of boundary nodes (A,B) = ($a_1, ..., a_k$ **,** $b_1, ..., b_k$) such that the entire sequence ($a_1, ..., a_k$ **,** $b_1, ..., b_k$) is in circular order, will be called a circular pair. Note that in section 5, we will want to seperate (or divide) the circular pair (A,B) by a set of intervals denoted (i_i ,i_j) such that$i \neq j$ and$i$ ¡ j. This notion will be clear later on.

A circular pair (A,B) of boundary nodes is said to be connected through Γ if there are k disjoint paths $\alpha_1, ..., \alpha_k$ in Γ, such that α_i starts at a_i, ends at b_i, and passes through no other boundary nodes. We say that α is a connection from A to B.

For each circular planar graph Γ, let $\pi(\Gamma)$ be the set of all circular pairs (A,B) of boundary nodes which are connected through Γ.

Recall there are two ways in which we can remove an edge from a graph Γ. First, we can delete an edge. Second, we can contract an edge to a single node. (An edge joining two boundary nodes is not allowed to be a contracted to a single node.)

We say that removing an edge breaks the connection from A to B if there is a connection from A to B through Γ, but there is not a connection from A to B after the edge is removed. A graph Γ is called **critical** if the removal of any edge breaks some connection in $\pi(\Gamma)$. *The final result of this paper is to produce a critical graph including all interior nodes and edges by simply gathering all necessary information at the boundary of the graph.* Think of a "fortune teller" predicting the shape of an object concealled within a foggy crystal ball by simply feeling the texture of its surface.

A graph Γ remains critical under Y - Δ equivalence transformations. Briefly, a Y - Δ equivalence is a geometric transformation shown below which maintains electrical equivalence since we replace three edges by three edges. For more information regarding the properties of Y - Δ equivalences in Γ, please see [1]. Y - Δ equivalence transformation in Γ A conductivity on a graph Γ is a function γ which assigns to each edge $e \in$ E a positive real number γ(e). A resistor network (Γ,γ) consists of a graph with a boundary together with a conductivity function γ. This paper makes no attempt to recover conductivities from boundary measurements. Therefore, we will not talk much about conductivities, except in the examples which conclude this paper. However, it should be noted that there is a linear map from boundary functions to boundary functions defined as follows. For each voltage potential $f = \{f(v_i\ )\}$ defined at the boundary nodes, there is a unique extension of f to all the nodes of Γ which satisfies Kirchoff's current law,$\sum_{q \in \mathrm{N}(p)} \gamma(pq)(f(q) - f(p)) = 0$ where $N(p)$ represents all neighboring nodes to p and $p \in$V, and $q \in \partial$V orV. This function then gives a current $I = \{I(v_i\ )\}$ into the network at the boundary nodes. The linear map which sends f to I is called the Dirichlet-to-Neumann map and is represented by an $n \times n$ matrix denoted by Λ.**2. Medial Graphs.** We will investigate the key formula of this paper, namely, *R(A) = card(A) - black(A) - max(A,B)*, where *max(A,B)* is the rank of a particular submatrix within the Λ matrix and (A,B) represents the circular pairs as defined in section 1.

A medial graph$M$ is a circular planar graph such that its boundary nodes are 1-valent and its interior nodes are 4-valent. The name "medial" comes from the following construction that for each circular planar graph, Γ produces a corresponding medial graph M(Γ).

Suppose Γ = (V, E, ∂V) is a circular planar graph with n boundary nodes. Γ is assumed to be embedded in the closed unit disk **D**so that the boundary nodes v_1 ,..., v_n occur in clockwise order around a circle $C = \partial$**D**and the rest of Γ is in the interior of **D**. The medial graph M(Γ) depends on the embedding. First, for each edge e of Γ, let m_e be its midpoint. Next, place *2n* boundary points t_1 ,...,t_{2n} on C so thatt_1 ¡ v_1 ¡ t_2 ¡ t_3 ¡ v_2 ¡ ... ¡ t_{2n-1} ¡ v_n ¡ t_{2n} ¡ t_1 in the clockwise circular order around C.

(1) The vertices of M(Γ) consist of the points m_e for e $\in$ E, and the points t_i for i = *1,...,2n*.

(2) The edges in M(Γ) are as follows. Two vertices m_e and m_f are joined by an edge whenever e and f have a common vertex and e and f are incident to the same face in Γ. There is also one edge for each point t_j as follows. The point t_{2i} is joined by an edge to m_e where e is the edge of the form $e = v_i\ r$ which comes first after the arc $v_i\ t_{2i}$ in clockwise order around v_i . The point t_{2i-1} is joined by an edge to m_f where f is the edge of the form $f = v_i s$ which comes first after the arc $v_i\ t_{2i-1}$ in clockwise order around v_i .

The vertices of the form m_e of M(Γ) are 4-valent; the vertices of the form t_i are 1-valent.

An edge uv of a medial graph M has a direct extension vw if the edges uv and vw separate any other two edges incident to the vertex v. A path $u_0u_{1.}..u_k$ in M is called a geodesic arc if each edge $u_{i-1}u_i$ has edge u_iu_{i+1} as a direct extension. A geodesic arc $u_0u_1..u_k$ is called a geodesic if either

(1) u_0 and u_k are points on the circle C.

or

(2) $u_k = u_0$ and $u_{k-1}u_k$ has u_0u_1 as a direct extension. If each geodesic in M begins and ends on C, has no self-intersection, and if M has no lenses, we will say that M is lensless. For our purposes, we will only be looking at lensless graphs. For more information on lenses and various Lemmas associated to electrical equivalency of medial graphs with lenses, please see [2], section 4.1.2.

A triangle in M is a triple {*f,g,h*} of geodesics which intersect to form a triangle with no other intersections within the configuration.

Suppose {*f*,*g*,*h*} form a triangle. A motion of {*f*,*g*,*h*} consists of interchanging the configuration as shown below.**Lemma 4.1.1.** *Two circular planar graphs are Y-Δ equivalent if and only if their medial graphs are equivalent under motions.*

Proof. Each Y-Δ transformation of Γ corresponds to a motion on M(Γ). Conversely, a motion on M(Γ) corresponds to a Y-Δ transformation of Γ. ∎**3. Z-Sequences.**We begin this section with the study of the Z-sequence for a particular medial graph, M(Γ). And although we do not directly compute the Z-sequence within the computer algorithm, we mention it solely to provide a more detailed presentation of medial graphs.

Let Mbe a medial graph. Then Mwill have $n$geodesics each of which intersect Ctwice. The $n$geodesics intersect Cin *2n*distinct boundary points. These *2n*points are labelled $t_1, ...t_{2n}$, so thatt_1 ¡ t_2 ¡ t_3 ¡ ... ¡ t_{2n-1} ¡ t_{2n} ¡ t_1 are in circular order around C. The geodesics will be labelled as follows. Let g_1 be the geodesic which begins at t_1 . The remaining geodesicsare labelled $g_2, g_3, ..., g_n$ so that if i ¡ j, then the first point of intersection of g_i with C occurs before the first point of intersection of g_j with C in clockwise order starting from t_1 . For each i = *1,2,...,2n* let z_i be the number associated with the geodesic which intersects C at t_i . In this way we obtain a sequence z(M)= $z_1, z_2, ..., z_{2n}$, called the Z-sequence for M. Each of the numbers from 1 to n occurs in Z-sequence for M exactly twice. The tranformation above from left to right will be called unwinding between t_1 and t_2 . The inverse of this transformation, defined if the geodesics from t_1 and t_2 are different and do not intersect in a lensless graph, will be called winding between t_1 and t_2 . After winding or unwinding, the medial graph is still lensless and its Z-sequence changes by one transposition.

Lemma 4.1.6. *Two lensless medial graphs M_1 and M_2 are equivalent under motions if and only if the Z-sequence of M_1 equals the Z-sequence of M_2 .*

Proof. Obviously, motions of a medial graph do not change its Z-sequence.

We show the other direction by an induction on the number of interior nodes of the medial graphs. Clearly, the lemma is true if M_1 or M_2 have no interior vertices. Now, suppose tghey have at least one. Then not all geodesics in M_1 or M_2 are parallel. WLOG we can assume that none of the geodesics of M_1 or M_2 terminate at two adjacent boundary nodes, that is there are no two equal adjacent symbols in the Z-sequence of M_1 or the Z-sequence of M_2 . Therefore, WLOG we can assume that the geodesics that go through boundary nodes 1 and 2 intersect in an interior vertice p_i in M_i , i = *1,2*. By a finite sequence of motions all other geodesics can be moved out of the triangle 1,2,p_i. Therefore, WLOG the medial graphs look like the following figure near the boundary vertices 1 and 2. The unwinding transformation above produces two new lensless medial graphs with equal Z-sequences. By the inductive statement, since these new medial graphs have fewer interior vertices, they are equivalent


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A motion of f,g,h in the Medial Graph

h                   h
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under motions, and therefore, so are the original graphs. ▪**4. Connections and Z-sequences. Key Identity.** Let Γ be a circular planar graph. A *path* βbetween boundary nodes $a$and $b$of Γ is either and edge (ab) or a sequence of interior nodes $p_1, ..., p_m$ such that$(ap_1), (p_1p_2), ..., (p_{m-1}p_m), (p_mb)$ are edges of Γ.

A disjoint connection α between two disjoint k-tuples of boundary nodes $a_1, ..., a_k$ and $b_1, ..., b_k$ is a set of pairwise disjoint paths α_i between the $a_i's$ and $b_i's$.

The following theorem, proved in [1], shows that the existense of disjoint connections between non-interlacing k-tuples of boundary nodes of Γ on C can be read directly from a Dirichlet-to-Neumann map Λ.

Theorem 4.1.7. (see [1]) *Let $a_1, ..., a_k$ and $b_1, ..., b_k$ be a disjoint pair of non-interlacing boundary nodes of Γ. Then there is a disjoint connection between the $a_i's$ and $b_i's$ if and only if* $\det\{\Lambda(a_i, b_j)\} \neq 0$. This states that the determinante of the submatrix in Λ formed by the rows a_i and the columns b_j is not equal to zero.

We now extend the notion of disjoint connections to medial graphs, M(Γ).

A face of medial graph M is a connected component of **D**- M. Due to the valences of the nodes in M one can color the faces of M in black and white so that no two faces with the same edge are of the same color (the so called two-coloring). If $M =$ M(Γ) then one can choose the two-coloring of M so that a face is black if and only if it contains a node of Γ. Let us call this coloring induced.

The boundary nodes of M split C in into *2n* intervals, namely, $i_1, i_2, ..., i_{2n}$. A two-coloring of M induces a two-coloring of the intervals.

For the remainder of this section, let c and d be two points in two distinct intervals i_k and i_j . Let C - $\{c,d\}$ = $A \cup B$ where A and B are connected disjoint geodesic arcs. Let I and J be two black intervals on the boundary such that $I \subset A$ and $J \subset B$. A path G between I and J is a sequence of black faces $F_1, ..., F_m$ such that $I \in F_1$, $J \in F_m$, $F_i \cap F_{i+1} \neq \emptyset$, $F_2, ..., F_{m-1} \cap C = \emptyset$ and c and d are not in the closures of the $F_i's$.

Let i_i and J_i be two disjoint k-tuples of the black intervals, such that $i_i \subset A$ and $J_i \subset B$. A disjoint connection between the $i_i's$ and the $J_i's$ is a sequence of pairwise disjoint paths G_i between the $i_i's$ and the $J_i's$.

The definitions above are chosen so that the following lemma is true.

Lemma 4.1.8. *Let Γ be a circular planar graph. Suppose $M = M(\Gamma)$ is its medial graph with the induced coloring. Let $\{a_i\} \in A$ and $\{b_i\} \in B$ be two disjoint k-tuples of boundary nodes of Γ. Let i_i and J_i be corresponding black intervals. Then there is a disjoint connection between the $a_i's$ and the $b_i's$ if and only if there is a disjoint connection between the $i_i's$ and the $J_i's$.*