

DETERMINING THE CONDUCTORS IN A NETWORK WITH INTERIOR SOURCES OF CURRENT

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Abstract

This paper discusses the recovery of conductances in a square resistor network from measurements of boundary output currents generated by internal current sources. The F matrix is defined and its characteristics are described. An algorithm is developed and tested for solving the inverse problem. Results are given for these tests, which show significant improvement over previous results, and the properties of the algorithm are analyzed.

1 Introduction

We consider resistor networks similar to those described in [2] and [3] with modifications suggested by [1]. For every positive integer n , there is a square network Ω consisting of *nodes* and *edges* constructed as follows. The *nodes* of Ω are the integer lattice points (i, j) in the cartesian plane for $0 \leq i \leq n+1$ and $0 \leq j \leq n+1$ with the four corner points $(0, 0)$, $(n+1, 0)$, $(0, n+1)$, and $(n+1, n+1)$ excluded. The *edges* of Ω consists of all unit-length horizontal or vertical line segments connecting nodes in Ω . Two nodes are called *neighbors* if an edge connects them. All *interior* nodes of Ω have four neighbors, and all *boundary* nodes have one. Let $N(s)$ denote the set of all nodes which is a neighbor of node s . Throughout this paper s and t will be used to represent

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nodes of Ω in general. Interior and boundary nodes will be denoted by p and q respectively.

Let Ω_0 denote the set of all nodes in Ω . The set of all boundary nodes, numbered clockwise starting from the upper left, is denoted by $\partial\Omega_0 = \{q_1, q_2, \dots, q_{4n}\}$. The set of all interior nodes is denoted by $\text{int}\Omega_0 = \{p_1, p_2, \dots, p_{n^2}\}$. Thus, $\Omega_0 = (\partial\Omega_0 \cup \text{int}\Omega_0)$. Note that the numbering scheme for interior nodes is not consistent throughout this paper. An example of a network with $n = 5$ is shown in Figure 1.

	q_1	q_2	q_3	q_4	q_5	
q_{20}	p_1	p_2	p_3	p_4	p_5	q_6
q_{19}	p_{16}	p_{17}	p_{18}	p_{19}	p_6	q_7
q_{18}	p_{15}	p_{24}	p_{25}	p_{20}	p_7	q_8
q_{17}	p_{14}	p_{23}	p_{22}	p_{21}	p_8	q_9
q_{16}	p_{13}	p_{12}	p_{11}	p_{10}	p_9	q_{10}
	q_{15}	q_{14}	q_{13}	q_{12}	q_{11}	

Figure 1

The set of edges is denoted by Ω_1 , where each edge represents a conductor. Individual edges are labelled according to which two nodes they join, i.e. $p_i p_j$ or $p_i q_j$ or st . The function γ which assigns a positive real-value to each edge in Ω_1 is called the *conductivity*. Let γ_i denote the conductivity of the conductor joining q_i to its interior neighbor, $\gamma(q_i p_j)$. Similarly, let $\gamma_{i,j}$ denote the conductivity between p_i and p_j , $\gamma(p_i p_j)$. A *network of resistors* is a network $\Omega = (\Omega_0, \Omega_1)$ with a γ on Ω_1 . Let u be a function on Ω_0 representing the potential at each node. Let $I(s)$, a function on Ω_0 , denote the current at each node s in Ω_0 . The only condition on the potential function u is Kirchoff's Current Law which states that

$$\sum_{t \in N(s)} \gamma(st)(u(s) - u(t)) = I(s) \quad (1)$$

for all s in Ω_0 . The function u is called γ -harmonic at all nodes s where

$I(s) = 0$. Thus, u is γ -harmonic at all interior nodes except at those specified as sources of internal current.

Throughout this paper the potential at all boundary nodes is assumed to be zero.

2 Preliminary Properties

2.1 The Kirchhoff matrix K

The square Kirchhoff matrix K is defined in Section 3 of [4]. It has the following interpretation. If u is a voltage matrix defined at the nodes of Ω , then Ku is the resulting current flow. The definition of K implies that it is symmetric. [4] also shows that K is positive semi-definite.

We assume the numbering scheme and block structure used in Theorem 3.2 of [4]. Thus K has the following block structure,

$$K = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}. \quad (2)$$

By the definition of K , and since all boundary potentials are zero,

$$\begin{pmatrix} A & B \\ B^T & D \end{pmatrix} \begin{pmatrix} 0 \\ \psi \end{pmatrix} = \begin{pmatrix} J \\ e \end{pmatrix}, \quad (3)$$

where ψ is the vector of interior potentials, J is the vector of boundary currents, and e is the vector of interior currents.

2.2 The F matrix

Definition 1. For $1 \leq i \leq 4n$ and $1 \leq j \leq n^2$, define each entry $f_{i,j}$ in F to be the current flow out of boundary node q_i due to a source current of +1 at p_j .

As an immediate consequence of Definition 1, if e is the vector of current flowing through each interior node, then Fe represents the vector of current, $-J$, flowing out of the boundary nodes. That is,

$$Fe = -J. \quad (4)$$

There are two simple properties of F which can now be shown. Both will become useful in later sections.

Lemma 2. *The sum of the entries in each column of F is 1.*

Proof. Fe_i produces a vector representing the total current flowing out of the network due to a source current of +1 at p_i . Since the total current out of a network must equal the total current into the network by Kirchhoff's Current Law, the sum of the entries in Fe_i , and therefore the sum of the entries in the i^{th} column of F , must equal 1. \square

Lemma 3. $F = -BD^{-1}$.

Proof. By Equation (3),

$$B\psi = J, \quad \text{and} \quad D\psi = e.$$

By Lemma 3.1 in [4], D is nonsingular and thus invertible. Therefore,

$$BD^{-1}e = J,$$

and by Definition 1, the lemma is proved. \square

Lemma 3 expresses F in terms of blocks in K . The next section will show that F is thus related to the U matrix.

2.3 The U matrix

Definition 4. *Let $u_i(j)$ represent the potential at interior node p_j due to a source current of +1 at p_i and of 0 everywhere else in the interior.*

Definition 5. *Let u_i represent the vector of interior potentials due to a single interior source current of +1 at p_i ,*

$$u_i = \begin{pmatrix} u_i(1) \\ u_i(2) \\ \cdot \\ \cdot \\ u_i(n^2) \end{pmatrix},$$

and let U be the $(n^2 \times n^2)$ matrix $[u_1, u_2, \dots, u_{n^2}]$.

Lemma 6. $DU = I$.

Proof. Since u_i represents a vector of interior potentials, from Equation (3)

$$Du_i = e_i,$$

where e_i is the vector of interior currents representing a current source of +1 at p_i and of 0 everywhere else in the interior. Numerically, e_i is the i^{th} column of the $(n^2 \times n^2)$ identity matrix. As a result,

$$DU = I \quad \text{and} \quad U = D^{-1}.$$

□

Corollary 7. U is symmetric. Thus, $u_i(j) = u_j(i)$.

Proof. Since K is symmetric, D is also symmetric. Thus,

$$D = D^T, \quad \text{which implies} \quad D^{-1} = (D^{-1})^T.$$

Therefore, from Lemma 6, U is also symmetric. □

Having defined U , we can express F in terms of γ_k and $u_i(j)$,

$$F = \begin{pmatrix} \gamma_1 u_1(1) & \gamma_1 u_2(1) & \cdots & \gamma_1 u_{n^2}(1) \\ \gamma_2 u_1(2) & \gamma_2 u_2(2) & \cdots & \gamma_2 u_{n^2}(2) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_n u_1(n) & \cdot & \cdots & \cdot \\ \gamma_{n+1} u_1(n) & \cdot & \cdots & \cdot \\ \gamma_{n+2} u_1(n+1) & \cdot & \cdots & \cdot \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{2n} u_1(2n-1) & \cdot & \cdots & \cdot \\ \gamma_{2n+1} u_1(2n-1) & \cdot & \cdots & \cdot \\ \gamma_{2n+2} u_1(2n) & \cdot & \cdots & \cdot \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{3n} u_1(3n-2) & \cdot & \cdots & \cdot \\ \gamma_{3n+1} u_1(3n-2) & \cdot & \cdots & \cdot \\ \gamma_{3n+2} u_1(3n-1) & \cdot & \cdots & \cdot \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{4n-1} u_1(4n-4) & \gamma_{4n-1} u_2(4n-4) & \cdots & \gamma_{4n-1} u_{n^2}(4n-4) \\ \gamma_{4n} u_1(1) & \gamma_{4n} u_1(1) & \cdots & \gamma_{4n} u_{n^2}(1) \end{pmatrix}. \quad (5)$$

Another useful property concerning the potentials in a network with interior sources of current follows.

Theorem 8. *The Maximum Principle. If there is a single interior current source at node p , the potential function u being γ -harmonic elsewhere, and all boundary potentials are zero, then the magnitude of the potential at the source node is larger than the magnitude of the potential at any other node.*

Proof. Consider the interior current source node, p , as a boundary node, and thus u is γ -harmonic everywhere in the interior. Corollary 2.2 in [2] states that the magnitude of all interior potentials would then be less than or equal to the magnitude of the potential at p . The potential at p is non-zero because otherwise no current would flow through the network by Corollary 2.3 in [2], but we assume there is at least one current source. Since the potential at p is non-zero, all interior potentials have lesser magnitudes than at p . Otherwise all potentials throughout the network must be equal, which is contradictory since the potential at some boundary nodes is zero and at others they're non-zero. \square

The following is now stated as a proposition, but follows from [1] and will be shown true later in this paper.

Proposition 9. *If a square resistor network produces an F matrix and diagonals of U , no other square network can produce the same F matrix and U diagonals.*

3 Relations in F

Theorem 10. *Kirchhoff's Condition on U is expressed by $DU = I$.*

Proof. Equation (5) shows that F is expressible in terms of boundary conductors and entries in U . The conductors depend on the network. The entries of U depend on Equation 1. Thus, if all the neighbors of p_j are in the interior, then for $j \neq i$

$$\sum_{p_k \in N(p_j)} \gamma(p_j p_k) (u_i(j) - u_i(k)) = 0,$$

and for $j = i$

$$\sum_{p_k \in N(p_j)} \gamma(p_j p_k)(u_i(j) - u_i(k)) = 1.$$

If some neighbor of p_j is a boundary node, then for $j \neq i$

$$\gamma(p_j q_j)u_i(j) + \sum_{p_k \in N(p_j)} \gamma(p_j p_k)(u_i(j) - u_i(k)) = 0,$$

and for $j = i$

$$\gamma(p_j q_j)u_i(j) + \sum_{p_k \in N(p_j)} \gamma(p_j p_k)(u_i(j) - u_i(k)) = 1.$$

By the definition of K in [4], if $i \neq j$, then $D_{i,j} = -\gamma(p_i p_j)$ if $p_i \in N(p_j)$, else $D_{i,j} = 0$. If $i = j$, then $D_{i,i} = \sum_{t \in N(p_i)} \gamma(p_i t)$. Now notice that the entries of DU are of the form $\sum_{t \in N(s)} \gamma(st)(u_i(s) - u_i(t))$. Therefore each equation expressed by $DU = I$ is a condition of Equation 1. \square

Note that the condition on F stated by Lemma 2 also follows from the equations in $DU = I$, particularly $Du_i = e_i$ for $1 \leq i \leq n^2$. Observe that the sum of the entries in i^{th} column of F equals the sum of rows in Du_i . Both these sums equal the sum of the entries in e_i , which is 1.

Another consequence of $DU = I$ is that U is symmetric (Corollary 7). By Lemma 3, $F = -BU$. So although F is not symmetric, it has symmetric characteristics due to U . For example,

$$f_{1,2}/f_{2,1} = \gamma_1/\gamma_2.$$

3.1 F and Non-Intersecting Connections in a Network

Let $P = (p_1, p_2, \dots, p_k)$ be a sequence of k distinct interior nodes and $Q = (q_1, q_2, \dots, q_k)$ be a sequence of k distinct boundary nodes such that there is only one permutation of non-intersecting connections between P and Q . That is, if there is a connection between P and Q , it must connect p_i to q_i for $1 \leq i \leq k$.

Lemma 11. *If (1) all boundary nodes have a potential of zero, (2) all interior nodes not in the set P have currents of zero, and (3) the currents at the boundary nodes of Q are all zero, then the potential at all nodes is zero.*

Proof. Consider a network similar to that above except all nodes in P are treated as boundary nodes. Then Theorem 4.1 in [4] states that $\det\Lambda(Q; P) \neq 0$ due to the conditions on P and Q . Thus the only solution to $\Lambda(Q; P)\vec{x} = 0$ is $\vec{x} = 0$. This implies that, in a network where the potential function is γ -harmonic in the interior and 0 at all boundary nodes not in P (still considered a boundary node), if the currents at the nodes in Q are all 0, then the potentials at the boundary nodes in P are all 0. Since the potentials at all boundary nodes are thus 0, by Corollary 2.3 in [2], all nodes have a potential of zero. \square

Theorem 12. *The determinant of a square matrix in F made up of rows labelled by Q and columns labelled by P , $\det F(Q, P)$, is non-zero.*

Proof. $\det F(Q, P) \neq 0$ if and only if the only solution to $F(Q, P)\vec{x} = 0$ is $\vec{x} = 0$. This is equivalent to stating that if the conditions described in Lemma 10 are all satisfied, then the currents at all nodes of P are 0. Such a statement is true since Lemma 10 shows that all the potentials in the network equal zero. Thus all the currents, including those at P , are 0, and the theorem is proven. \square

Because there are often many such permutations P and Q in square rectangular networks, Theorem 11 implies several relationships between the elements of F . For example, suppose it is possible to input certain amounts of current at specific interior nodes such that some boundary currents are zero. Consider Figure 2.

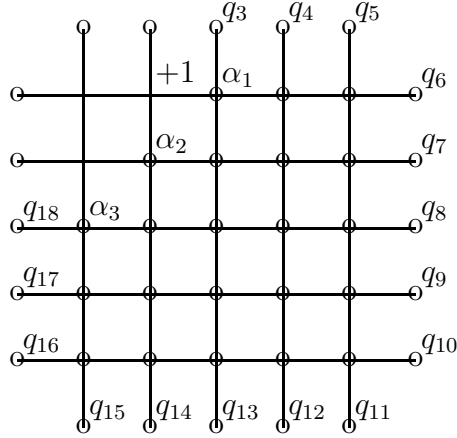


Figure 2

Suppose currents of $+1$, α_1 , α_2 , and α_3 are forced on the nodes as labelled such that the potential at all nodes marked by (o) is zero. Then it would be possible to express specific terms of F in a system of equations as follows. Let p_1, p_2, p_3, p_4 correspond to the nodes with currents $\alpha_1, \alpha_2, \alpha_3, +1$ respectively. Then,

$$f_{i,4} + \alpha_1 f_{i,1} + \alpha_2 f_{i,2} + \alpha_3 f_{i,3} = 0 \quad \text{for } 3 \leq i \leq 18$$

is true. Theorem 11 identifies which 3 equations can be used to solve for $\alpha_1, \alpha_2, \alpha_3$. Similar relations follow for different sets of interior nodes.

4 Theoretical Characterization of F in Simpler Networks

Up to this point we have identified the constraints on F , particularly $DU = I$. We have also identified three properties: (1) the sum of each column in F equals 1, (2) F is related to the symmetric matrix U , and (3) certain sub-matrices in F are non-singular (Theorem 11). The third property, for large networks, produces the most conditions on what the entries of F can be.

To simplify matters we shall now consider networks of the following type. For every positive integer n there is a *one-layer* network Γ with n boundary nodes and n interior nodes. Let Γ_0 denote the set of all nodes in Γ . The set

of interior nodes is denoted by $int\Gamma_0 = \{p_1, p_2, \dots, p_n\}$. Similarly, the set of boundary nodes is denoted by $\partial\Gamma_0 = \{q_1, q_2, \dots, q_n\}$. Let Γ be constructed as follows. Interior node p_1 is connected to p_n and p_2 , and p_n is connected to p_{n-1} and p_1 . In the same manner, for $1 < i < n$, p_i is connected to p_{i-1} and p_{i+1} . Also, each boundary node q_i is connected only to p_i . The cases for $n = 3$ and $n = 4$ are shown below.

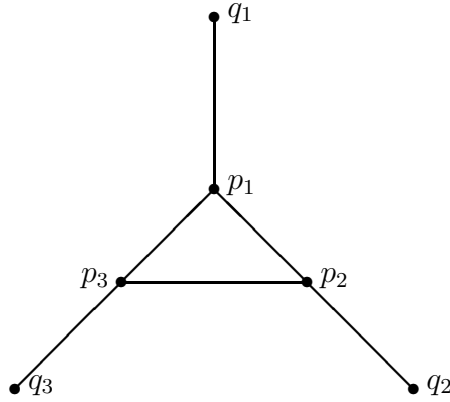


Figure 3

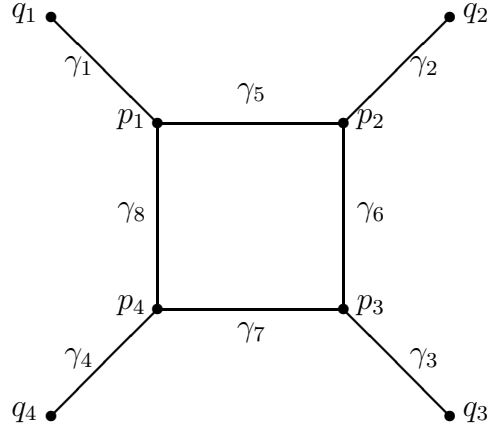


Figure 4

All other terms and conditions associated with this network are the same as described in Section 1.

4.1 Γ Network for $n = 4$

We shall describe two methods for recovering all of F using only minimal information from it. Note that knowing all of F is not sufficient for recovering the network. Thus we allow ourselves the option of knowing certain diagonal entries of the U matrix. [1] and later sections of this paper show how recovery of networks is possible from F and diagonals of U . Ideally, we can recover all m conductors of a network from knowing just m entries in F and in the diagonals of U . For example, in the case of $n = 4$ we should be able to recover all 8 conductors from 8 measurements. However, each equation in the system represented by $DU = I$ depends on interior conductors not immediate from F or U . Thus finding a solvable system of linear equations with only m measurements has proven difficult.

We now investigate Γ for $n = 4$, assuming the numbering scheme shown in Figure 4.

4.1.1 Method I

In this first method all the characteristics of F previously listed are utilized. Consider the following scheme for representing F .

$$F = \begin{pmatrix} \gamma_1 u_1(1) & \gamma_1 u_1(2) & \gamma_1 u_1(3) & \gamma_1 u_1(4) \\ \gamma_2 u_1(2) & \gamma_2 u_2(2) & \gamma_2 u_2(3) & \gamma_2 u_2(4) \\ \gamma_3 u_1(3) & \gamma_3 u_2(3) & \gamma_3 u_3(3) & \gamma_3 u_3(4) \\ \gamma_4 u_1(4) & \gamma_4 u_2(4) & \gamma_4 u_3(4) & \gamma_4 u_4(4) \end{pmatrix}. \quad (6)$$

We have made use of the symmetric property of U and thus reduced the number of variables in F by $4^2 - (4 + 10) = 2$. Since each column of F sums to 1, there are four additional relations. Now consider the following nine entries of F ,

$$F = \begin{pmatrix} 0 & * & 0 & 0 \\ * & 0 & * & 0 \\ * & * & 0 & * \\ * & * & * & 0 \end{pmatrix}.$$

Due to U symmetry, the boundary conductors and then the upper right triangle entries can be found. Using the column sum, the diagonal entries follow. So if we know $u_1(1)$ and those nine entries (*) of F , we can recover all of F and U , and subsequently all conductors in the network.

However, we still have not reduced the the number of measurements to eight. Perhaps Theorem 11 will supply the necessary two relations. Infact, the following equations does theoretical narrow the number of relations by two,

$$\begin{aligned} f_{2,1} + \alpha_1 f_{2,2} + \alpha_2 f_{2,4} &= 0 & f_{1,2} + \beta_1 f_{1,1} + \beta_2 f_{1,3} &= 0 \\ f_{3,1} + \alpha_1 f_{3,2} + \alpha_2 f_{3,4} &= 0 & f_{3,2} + \beta_1 f_{3,1} + \beta_2 f_{3,3} &= 0 \\ f_{4,1} + \alpha_1 f_{4,2} + \alpha_2 f_{4,4} &= 0 & f_{4,2} + \beta_1 f_{4,1} + \beta_2 f_{4,3} &= 0 \end{aligned}$$

but doesn't lead to a linear solution.

4.1.2 Method II

Consider $Du_1 = e_1$, written out as follows:

$$\begin{aligned} u_1(1)(\gamma_1 + \gamma_8 + \gamma_5) - u_1(2)(\gamma_5) - u_1(4)(\gamma_8) &= 1, \\ u_1(2)(\gamma_2 + \gamma_5 + \gamma_6) - u_1(3)(\gamma_6) - u_1(1)(\gamma_5) &= 0, \\ u_1(3)(\gamma_3 + \gamma_6 + \gamma_7) - u_1(4)(\gamma_7) - u_1(2)(\gamma_6) &= 0, \\ u_1(4)(\gamma_4 + \gamma_7 + \gamma_8) - u_1(1)(\gamma_8) - u_1(3)(\gamma_7) &= 0. \end{aligned}$$

Assume we can find γ_i and $u_1(i)$ for $1 \leq i \leq 4$. We are then left with a system of four equations and four unknowns. Unfortunately, this system has rank three and thus the interior conductors cannot be solved for. However, if the value of one interior conductor is found, the other conductors will follow from the system above.

To discover a conductor, say γ_5 , apply the following method. Put a source current of $+1$ at p_1 and a source current of α at p_4 such that the potential at p_1 and p_4 are equal. In other words,

$$u_1(1) + \alpha u_4(1) = u_1(4) + \alpha u_4(4).$$

Recall that we already know $u_1(1)$ and $u_1(4)$. To find α , it is also necessary to know $u_4(4)$. The current through γ_8 can thus be set to zero. We now recover the conductor γ_5 from Kirchhoff's Current Law applied at p_1 , namely

$$\gamma_1[(u_1(1) + \alpha u_4(1)) - (0) + \gamma_5[(u_1(1) + \alpha u_4(1)) - (u_1(2) + \alpha u_4(2))] = 1.$$

In addition to the previously mentioned, we must also know $u_4(2)$.

So, given $u_1(1)$ and the following eight values of F ,

$$F = \begin{pmatrix} 0 & * & * & * \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & * & 0 & * \end{pmatrix},$$

we can recover the *one-layer*, $n = 4$, network as follows. First determine $f_{1,1} = \gamma_1 u_1(1)$ by Lemma 2. Then, aware of U symmetry, use the complete first row and column of F to find the boundary conductors. $u_4(2)$ and $u_4(4)$ follows from $f_{4,2}$ and $f_{4,4}$, and then γ_5 subsequently. The other conductors can now be recovered from $Du_1 = e_1$. The remainder of F and U also follows from $DU = I$.

This method requires one measurement more than the number of conductors, but works well for all *one-layer* networks. A similar algorithm applicable to square rectangular networks is being investigated.

5 The Forward Problem

The forward problem is to find the matrix F for a particular network Ω , using the definition of F in Section 2. The calculation is simply a matrix

multiplication. The forward problem based on internal currents displays some attractive properties in comparison to the traditional forward problem of finding a map from boundary potentials to boundary currents. In particular, a random percentage change in the values of the conductors produces a comparable percentage change in the values of F , assuming conditions on the size of the network, the size of the change in the γ 's and the range of sizes of γ 's. In addition, the changes in F are less dependent on the size of the network and range of γ 's than one would expect. Some representative numerical results for this phenomenon are given in the following table. In the table, G^* is the vector of all γ 's with a consistent, but not given ordering, and F^* is the vector of all entries in the matrix F with a consistent, but not given ordering.

$$\Delta G = \left\| \frac{|G_r(i) - G_m(i)|}{G_r(i)} \right\|_{\infty}$$

and

$$\Delta F = \left\| \frac{|F_r(i) - F_m(i)|}{F_r(i)} \right\|_{\infty}$$

where $F_r(i)$ is a real value of F found using unchanged entries of G , and $F_m(i)$ is a modified value of F found using the modified entries of G , and likewise for all other vectors used in this paper.

n	Range of γ	ΔG	ΔF
8	$2 \leq \gamma \leq 3$	$1.0 * 10^{-6}$	$8.4845 * 10^{-5}$
		$1.0 * 10^{-4}$	$8.4849 * 10^{-3}$
		$1.0 * 10^{-2}$	0.85297
		1.0	0.72056
8	$1 \leq \gamma \leq 10$	$1.0 * 10^{-5}$	$4.5993 * 10^{-6}$
		$1.0 * 10^{-3}$	$4.6006 * 10^{-4}$
		0.1	$4.7330 * 10^{-2}$
		0.5	0.26019

6 Solution of the Inverse Problem

In this section, we describe the algorithm that we used to recover the conductors in a square resistor network, given the entries of F and the diagonal

entries of U . The algorithm is essentially a completion of the partial algorithm started in [1].

The algorithm is composed of three steps:

1. Recovery of Boundary Conductors
2. Two Current Conductor Recovery
3. Calculation of Potentials

6.1 Recovery of Boundary Conductors.

Let $\mu_j(i)$ be the current through exterior node q_i due to a current of 1 into p_j . Let $\mu_j(a, b)$ be the current through the edge connecting p_a and p_b due to an input current at p_j . Then the conductors $\gamma(i)$ for $1 \leq i \leq 4n$ can all be recovered using the formula:

$$\gamma(i) = \frac{\mu_{4n+i}(i)}{u_i(i)} = \frac{F_{i,4n+i}}{U_{i,i}} \quad (7)$$

The recovery of these conductors is both rapid and exact, as each calculation depends on only one value of F and one value of U .

6.2 Two Current Conductor Recovery

We use this procedure to recover all of the edges between interior nodes. The method uses two known conductors ($\gamma(b, e)$ and $\gamma(a, e)$ in Figure 5), along with potentials found using the method in Section 6.3 to recover two additional conductors ($\gamma(d, e)$ and $\gamma(c, e)$).

To recover $\gamma(d, e)$, we must know the current through $\gamma(d, e)$ and the potential difference between nodes d and e , for a given set of internal sources. We know that the current through $\gamma(d, e)$ will be equal to the current through all the other edges in Figure 5 plus any input current at node e . We will force a current of 0 through $\gamma(c, e)$. This is done by inserting a current of 1 at node e and a current of α at c such that the potential at e will equal the potential at c . This α is given by:

$$\alpha = \frac{u_e(e) - u_e(c)}{u_c(c) - u_e(c)}. \quad (8)$$

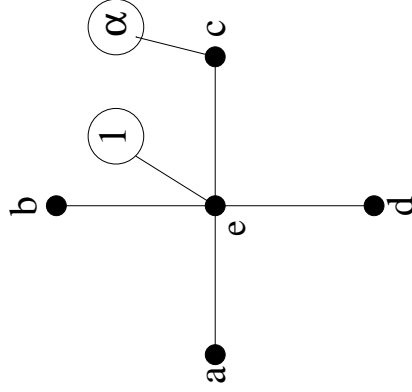


Figure 5

Then, the current through $\gamma(d, e)$ is:

$$\begin{aligned} \mu(d, e) = & 1 + \gamma(b, e)(\alpha u_c(b) + u_e(b) - u_e(e) - \alpha u_c(e)) \\ & + \gamma(a, e)(\alpha u_c(a) + u_e(a) - u_e(e) - \alpha u_c(e)) \end{aligned} \quad (9)$$

And

$$\begin{aligned} \gamma(d, e) &= \frac{\mu(d, e)}{pot(e) - pot(d)} \\ &= \frac{\mu(d, e)}{u_e(e) + \alpha u_e(c) - u_e(d) - \alpha u_c(d)} \end{aligned} \quad (10)$$

Similarly, by blocking the current between nodes d and e, we can also recover $\gamma(c, e)$. The placement of a sample structure such as in Figure 5 is shown below.

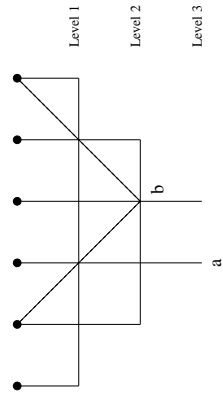


Figure 7

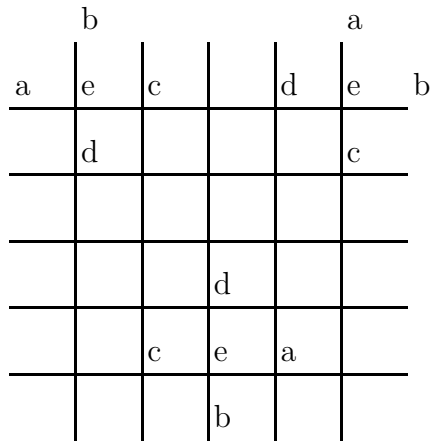


Figure 6

Remark 13. While the main purpose of blocking current through edge $c-e$ was to avoid calculations with an unknown value, this method also increases the current flowing through edge $d-e$, thus reducing the small-current problems found in former algorithms. It is also possible to insert currents at nodes a, b and c , such that the only current passing the edge $d-e$ is the current of 1 from node e . However, the errors introduced by an increased use of potentials and the solution of a system of three equations do not make up for the small increase in current through edge $d-e$.

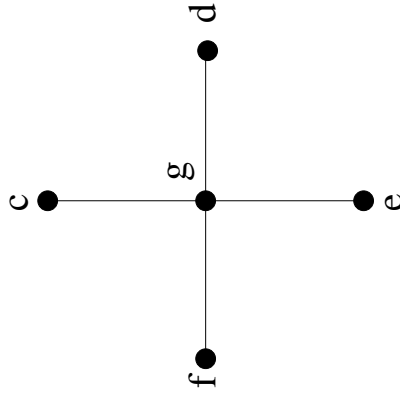


Figure 8

6.3 Calculation of Potentials

In order to make the calculations in the previous section, we required values of the potentials at nodes a,b,c and d due to an input current at e and the values of these potentials due to an input current at c or d. Suppose you wish to calculate $u_a(b) = u_b(a)$ in Figure 7, then

1. Determine which node, a or b is closer to a face of the network. In this case it is node b.
2. Determine which face node b is closest to.
3. Determine the nodes between b and this edge at which we must find the potentials due to an input current at a in order to find $u_a(b)$. These nodes will be those lying on or between the diagonal lines in Figure 7.
4. Calculate the potentials at level one using the formula $u_a(i) = F_{j,a}/\gamma(j)$ for boundary nodes q_j within the diagonal lines and interior nodes, p_i , on the first level and within the diagonal lines (level here is defined as in [1]).
5. Using these potentials and the conductors values $\gamma(f, g)$, $\gamma(c, g)$ and $\gamma(d, g)$ in Figure 8, calculate the currents $\mu(f, g)$, $\mu(c, g)$ and $\mu(d, g)$. These currents will equal $\mu(g, e)$, which we use with $u_a(g)$ and $\gamma(g, e)$ to find $u_a(e)$.
6. Finally, we repeat step 5 to determine the other potentials in Level 2 and all deeper levels until we find $u_a(b)$.

6.4 The Algorithm

These three procedures fit together in the following manner. First, use procedure 1 to recover the conductivity of all edges connected to boundary nodes. Second, use procedure 2, with values from procedure 3, to calculate the conductivity of all edges in Level 1 and connecting Level 1 to Level 2. We

accomplish this by using the procedure on the North side, then rotating the structure in Figure 5 clockwise 90 degrees and doing the East side, then rotating for the South and again for the West. (The North side is the set of nodes closest to exterior nodes q_1 through q_n , the East is the set of nodes closest to nodes q_{n+1} through q_{2n} , etc.) This is illustrated in Figure 9 which shows all b and e nodes for Level 1. A similar process is done to recover all other levels. Level 2 is illustrated in Figure 10.

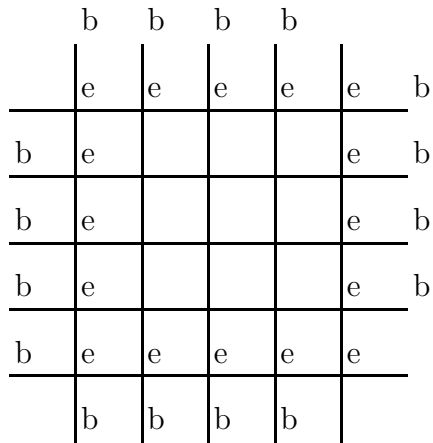


Figure 8

		b	b	b	
	b	e	e	e	b
	b	e		e	b
	b	e	e	e	b
		b	b	b	

Figure 9

7 Amount of Information Required

While we assumed the knowledge of all of the F and the diagonal of U , this is far more information than the number of conductors and all of it should not be necessary. In fact, the algorithm does not use all of the information we make available to it.

The intermediate information that we need is all in the form of potentials. Therefore, the best way to determine how much information is required is to determine which potentials are required and which values of F are necessary to calculate those potentials. But first, let us look at the diagonal entries of U . All of the diagonal entries of U are required by the algorithm described in Section 6, because every internal node is used in one unit, such as that in Figure 5, as the center node, e . $u_e(e)$ is required by each step 2 and therefore all diagonal values of U are required.

Now we turn to F ,

Definition 14. *A quadrant of a square network is the set of nodes such that each node in the quadrant is closest to a single face. If a node is equidistant from 2 or more faces, it will belong to both respective quadrants. Thus we will speak of the North, East, South and West quadrants. A quadrant is illustrated*

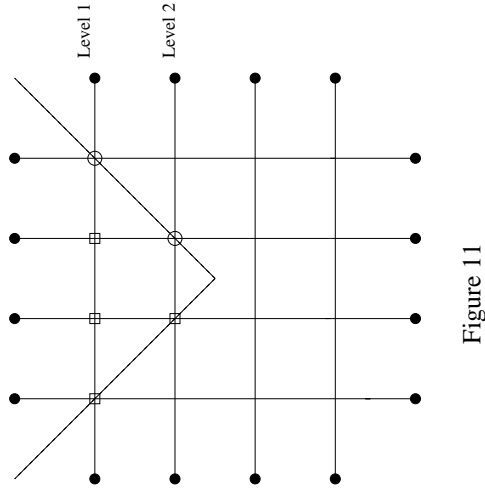


Figure 11

in Figure 11 in which the set of nodes on and between the diagonal lines form a quadrant.

In our algorithm, given a node denoted by c for this analysis, we require at most the potentials at all adjacent nodes to c , as well as the potentials at the nodes two edges away in a horizontal or vertical direction. Knowing this, we can find the set of all necessary potentials in the following way: Take the set of c 's as all nodes in a given quadrant except along the right diagonal border of the quadrant (right diagonal border nodes are marked with circles in Figure 11), and find the potentials, due to a current of 1 put into these nodes, at the nodes one and two edges to the right of them, hereafter referred to as u_r and u_{rr} respectively. In the process of finding u_a (the potential at the node above c , due to a current at c), we must also find u_{aa} (the potential two nodes above c). In addition, in the process of finding u_r , we must use the nodes immediately above, diagonally to the above and right, and diagonally to the above and left. The node diagonally to the above and left is u_a , thus we will find both u_a and u_{aa} in the process of finding u_r . u_l and u_b (left and below) will be found by the symmetry of U .

On the right diagonal boundary, find the potential at the node below c due to a current at the node above c (u_{ab}). In addition, because we are not calculating u_r , we must calculate u_a directly. After dealing with all four quadrants, the potentials calculated to this point will be sufficient for our algorithm.

Finding u_r and u_{rr} requires $2l$ values of F where l is the Level of node c .

Finding u_{ab} and u_a along the right diagonal boundary each requires $1+2(l-2)$ values of F , giving a total of $2(2l-3) = 4l-6$. Note that this calculation is only done in levels greater than one. In each level of a quadrant there are $(n-1) - 2(l-1)$ non-boundary nodes and one node on the right boundary. There are $\lceil (n+1)/2 \rceil$ levels in any network. Therefore, the total number of values from F that are used to find all the potentials for the algorithm is

$$4 \left(\sum_{l=1}^{\lceil (n+1)/2 \rceil} (2ln - 4l^2 + 2l) + \sum_{l=2}^{\lceil (n+1)/2 \rceil} (4l - 6) \right). \quad (11)$$

An additional $4n$ values of F are required for the boundary nodes. The following table shows some values of Equation 11 for various sizes of networks, along with the total number of entries in F and the number of conductors in the network:

n	<i># of Edges</i>	<i># of F Entries Used</i>	<i>Total Entries in F</i>
3	24	28	108
8	144	344	2048
10	220	608	4000
12	312	976	6912
16	554	2088	16384
20	840	3808	32000
32	2112	13896	131072

Initially, the number of measurements that are required is approximately equal to the number of conductors. As the size of the network increases, the ratio of information to conductors becomes larger. Luckily, however, the growth of required measurements is far slower than the growth in the total number of values of F .

8 Results of Numerical Tests

As expected, using internal sources produced improved results over using only boundary sources. Assuming complete double precision accuracy (16 digits) in the given information, we were able to recover square networks up to size $n=24$. Beyond $n=24$ we were unable to store and work with the required matrices without sparse matrix techniques. We did not have time

to implement these techniques, but we believe that the algorithm would have recovered networks of at least $n=30$. We also performed tests using values of F that included random errors, such as would exist in a physical situation. Results on these tests were also favorable, as we were able to recover networks with reasonable accuracy using values of F that included errors within the range of real measuring devices. The results are given and explained in the following sections.

8.1 Zero Error Tests

The following table gives the results we found assuming double precision accuracy in the given values of F and the diagonals of U . The execution times were all reasonable, particularly for the smaller networks and the errors showed a steady increase with increased network size. The γ 's that we used were randomly distributed within the range of 2 to 3.

<i>Size of n</i>	<i>Time to Run</i>	ΔG
4	1sec.	$2.978D - 15$
8	2	$1.390D - 13$
12	8	$6.473D - 12$
16	2 : 05	$2.359D - 10$
20	3 : 10	$9.289D - 8$
24	6 : 00	$6.357D - 6$

A lot of the lost accuracy in these tests is due to roundoff error that is covered by larger errors in tests which assume less accuracy in F . Therefore, the loss in accuracy will not agree with the results of the following section.

8.2 Tests With Assumed Error

The tables in the appendix give the results of our tests with errors assumed in F . We noted two major features of the algorithm from these results. First, for small errors in the entries of F , the errors in the recovered γ 's were directly proportional to the errors in the entries of F . This feature will be proven in the next section. Second, beyond a threshold value of error in F , the errors in recovered γ 's became highly erratic. This is due to the errors accumulated in the third procedure and will also be examined in the following section.

9 Analysis of Errors

Theorem 15. *Assuming perfect accuracy in the calculation of potentials,*

$$\|\gamma_{d,e}(r) - \gamma_{d,e}(c)\|_\infty = k_1 \|\gamma_{a,e}(r) - \gamma_{a,e}(c)\|_\infty + k_2 \|\gamma_{b,e}(r) - \gamma_{b,e}(c)\|_\infty \quad (12)$$

where k_1 and k_2 are constants, $\gamma(r)$ is a real γ and $\gamma(c)$ is a calculated γ .

Proof. The error in the current, $\mu(e, d)$, is proportional to the sum of constant multiples of the errors in input conductors, assuming no error in potentials, by Equation 9. The error in $\gamma(d, e)$, as calculated in Equation 10, depends on a constant multiple of this current and therefore on a sum of constant multiples of the errors in input conductors. \square

This theorem leads us to believe that the linear dependence of errors in calculated γ 's before the breakdown point is due to the errors in the previous γ 's used in step two of the algorithm. This claim is further supported by the fact that the relative errors in the potentials are typically at least one to two orders of magnitude smaller than the relative errors in the input γ 's below the breakdown point, and become equal to or greater than the errors in the input γ 's as we move beyond the breakdown point. This also explains the unpredictable behavior of the algorithm beyond the breakdown point. The calculated potentials are factored into the second procedure in a multitude of ways and would thus interact unpredictably, sometimes cancelling, sometimes adding, to produce unpredictable results.

10 Appendix

n	$range\ of\ \gamma$	ΔF	ΔG
8	2..3	$1.0 * 10^{-6}$	$1.1245 * 10^{-5}$
		$1.0 * 10^{-4}$	$1.1238 * 10^{-3}$
		$1.0 * 10^{-2}$	0.10488
		$3.0 * 10^{-2}$	0.30398
		$3.1 * 10^{-2}$	1.4969
		$3.2 * 10^{-2}$	0.95171
		1.0	80.338
		2.0	$9.749 * 10^{+15}$
8	1..10	$1.0 * 10^{-5}$	$2.5692 * 10^{-3}$
		$1.0 * 10^{-4}$	$2.5640 * 10^{-2}$
		$1.0 * 10^{-3}$	0.25293
		$5.0 * 10^{-3}$	1.4032
		$1.0 * 10^{-2}$	9.1905
		0.1	289.65
12	2..3	$1.0 * 10^{-5}$	$3.0135 * 10^{-2}$
		$1.0 * 10^{-4}$	0.29089
		$7.5 * 10^{-4}$	1.6702
		$1.0 * 10^{-3}$	1.2125
		$1.5 * 10^{-3}$	13.022
		$1.0 * 10^{-2}$	592.226
12	1..10	$1.0 * 10^{-7}$	$6.3097 * 10^{-2}$
		$1.0 * 10^{-6}$	0.37217
		$1.0 * 10^{-5}$	3.1208
		$1.0 * 10^{-4}$	65.5581
16	2..3	$1.0 * 10^{-8}$	$2.8734 * 10^{-3}$
		$1.0 * 10^{-7}$	$2.8692 * 10^{-2}$
		$1.0 * 10^{-6}$	0.28135
		$7.5 * 10^{-6}$	36.838
		$9.0 * 10^{-6}$	4.8142
		$1.0 * 10^{-5}$	111.75
16	1..10	$1.0 * 10^{-10}$	$1.381 * 10^{-3}$
		$1.0 * 10^{-9}$	$1.4040 * 10^{-2}$
		$1.0 * 10^{-8}$	0.16742
		$5.0 * 10^{-8}$	5.3461
		$8.0 * 10^{-8}$	0.89297
		$1.0 * 10^{-7}$	1.2960
		$1.0 * 10^{-6}$	274.19

References

- [1] C. M. Cook, A. V. Iglesias, and L. M. Judd, *Locating a Diode in a Rectangular Network Using Internal Current Sources*.
- [2] E. B. Curtis and J. A. Morrow, *Determining the resistors in a network*, SIAM J. Appl. Math., 50 (1990), pp. 918-930.
- [3] E. B. Curtis and J. A. Morrow, *The Dirichlet to Neumann map for a resistor network*, SIAM J. Appl. Math., 51 (1991), pp. 1011-1029.
- [4] E. B. Curtis, D. Ingreman, and J. A. Morrow, *Circular Planar Graphs and Resistor Networks*.