Combining Critical Graphs

Erika L. Shubert Michael S. McLendon *

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Abstract

We will consider connected circular planar graphs. We combine two critical c.c.p. graphs Γ_1 and Γ_2 by identifying k boundary nodes from Γ_1 with k boundary nodes from Γ_2. The combined graph is denoted $\Gamma_1 \vee \Gamma_2$ and may or may not be critical. We use the z-sequences of Γ_1 and Γ_2 to find the z-sequence of a critical graph Γ that has the same set of connections as $\Gamma_1 \vee \Gamma_2$. We describe an algorithm to find the z-sequence of Γ and implement this algorithm in a computer program.

1 Introduction

This article was inspired by [3] in which Rosema explores combinations of resistor networks using Dirichlet-Neumann maps. A resistor network is a graph with an associated conductivity function. We will examine combinations of circular planar graphs using techniques developed by Curtis, Ingerman, and Morrow in [2] and Colin de Verdière, Gitler, and Vertigan in [1].

A graph $\Gamma = (V, E)$ consists of nodes $V = \{v_1, v_2, ..., v_n\}$ and edges $E = \{e_{ij}\}$ such that e_{ij} connects node v_i to node v_j. The set of nodes consists of two subsets: a set of boundary nodes V_B and a set of interior nodes V_I. A connected circular planar (c.c.p.) graph is a graph embedded in the plane such that the boundary nodes lie on the circle C and the interior nodes and all edges are inside C.

*e-mail: shubert@wittenberg.edu, msm5@ra.msstate.edu

The boundary nodes of a c.c.p. graph Γ are numbered either clockwise or counterclockwise on C. A sequence $w_1, w_2, ..., w_m$ is in circular order if some cyclic permutation is in numerical order. A pair of sequences of boundary nodes $(P; Q) = (p_1, ..., p_k; q_1, ..., q_k)$ such that the entire sequence $(p_1, ..., p_k, q_k, ..., q_1)$ is in circular order is a circular pair.

A circular pair $(P; Q)$ of boundary nodes is connected through Γ if there are k disjoint paths $\alpha_1, ..., \alpha_k$ in Γ, such that α_i starts at p_i, ends at q_i, and passes through no other boundary nodes. The set of paths α is a connection from P to Q. For each c.c.p. graph Γ, $\pi(\Gamma)$ is the set of all circular pairs $(P;Q)$ of boundary nodes which are connected through Γ.

If the removal of any edge in a graph Γ breaks some connection in $\pi(\Gamma)$, then Γ is a critical graph.

Each c.c.p. graph Γ has an associated medial graph, $\mathcal{M}(\Gamma)$. $\mathcal{M}(\Gamma)$ consists of vertices (interior and boundary) and medial edges which connect the vertices. $\mathcal{M}(\Gamma)$ is formed in three steps:

1. place two boundary vertices on C between every two boundary nodes of Γ,
2. place interior vertices at the midpoint of each edge in Γ,
3. connect the vertices with medial edges.

The placement of the medial edges is restricted as follows:

1. no two boundary vertices may be connected,
2. connect each boundary vertex in $\mathcal{M}(\Gamma)$ to exactly one interior vertex (boundary vertices are one-valent),
3. connect each interior vertex to exactly four vertices (interior vertices are four-valent),
4. a medial edge cannot cross an edge in Γ.

Figure 1.1 shows a graph Γ and its associated medial graph, $\mathcal{M}(\Gamma)$. The heavy lines are the edges of Γ. A node of Γ is represented by a filled circle ($\bullet$). The thin lines are the medial edges of $\mathcal{M}(\Gamma)$.

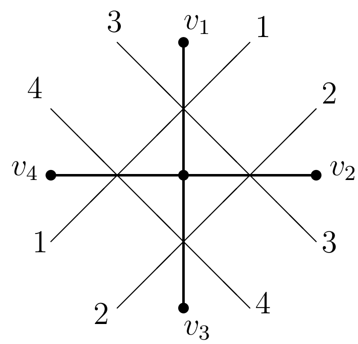

FIGURE 1.1

Given an interior vertex v in $\mathcal{M}(\Gamma)$, a medial edge uv has a direct extension vw if the medial edges uv and vw separate the two other medial edges incident to v. A path of medial edges $u_0u_1, u_1u_2, u_2u_3, ..., u_{k-1}u_k$ in $\mathcal{M}(\Gamma)$ is a geodesic arc if each medial edge $u_{i-1}u_i$ has medial edge u_iu_{i+1} as a direct extension. A geodesic arc $u_0u_1, u_1u_2, u_2u_3, ..., u_{k-1}u_k$ is a geodesic if either

1. u_0 and u_k are boundary vertices, or

2. $u_k = u_0$ and $u_{k-1}u_k$ has u_0u_1 as a direct extension.

A geodesic forms a loop if it begins and ends at the same vertex. If a geodesic intersects itself, it forms a self-intersection. If two distinct geodesics intersect at two different vertices, they form a lens.

A graph Γ has an associated z-sequence. The z-sequence is formed by numbering each boundary vertex in $\mathcal{M}(\Gamma)$ such that when two vertices are connected by a geodesic, those two vertices have the same number. If Γ has n boundary nodes, then the z-sequence is a sequence of the numbers from 1 to n where each number occurs exactly twice. The z-sequence for the medial graph in Figure 1.1 is $z = \{1, 2, 3, 4, 2, 1, 4, 3\}$.

To form the z-sequence of Γ each geodesic in $\mathcal{M}(\Gamma)$ that intersects the boundary circle is labeled. For simplicity, we refer to a geodesic in $\mathcal{M}(\Gamma)$ using its corresponding label in the z-sequence.

A boundary node p in a c.c.p. graph Γ_1 is identified with a boundary node

q in a c.c.p. graph Γ_2 by replacing both p and q with a single node, $p = q$. Given $s \in V_B$, s is interiorized by changing it from a boundary node to an interior node.

A geodesic g in $\mathcal{M}(\Gamma_1)$ is joined with a geodesic h in $\mathcal{M}(\Gamma_2)$ by "identifying" a boundary vertex of g with a boundary vertex of h, forming one geodesic from g and h.

Given four nodes s, p, r, q and three edges ps, rs, qs (as in Figure 1.2a), a $Y - \Delta$ transformation removes the node s and the edges ps, rs, qs and adds three new edges pq, qr, rp (as in Figure 1.2b). A $\Delta - Y$ transformation reverses this operation. Two c.c.p. graphs Γ_1 and Γ_2 are $Y - \Delta$ equivalent if Γ_1 can be transformed to Γ_2 by a sequence of $Y - \Delta$ or $\Delta - Y$ transformations.

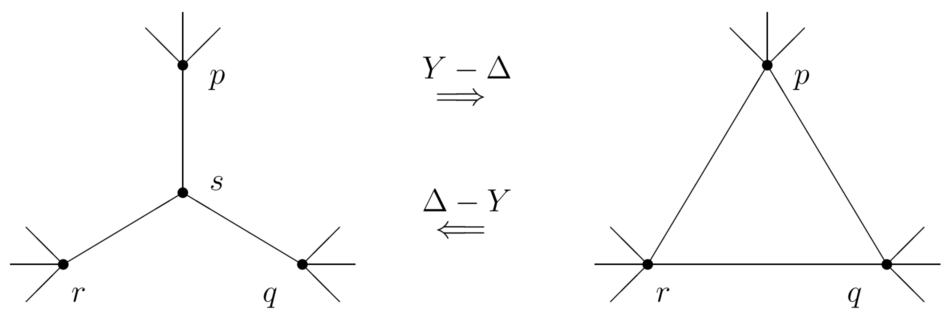

FIGURE 1.2a FIGURE 1.2b

A motion in $\mathcal{M}(\Gamma)$ moves a geodesic past the intersection of two other geodesics. Figure 1.3 shows a motion of the geodesic f past the intersection p.

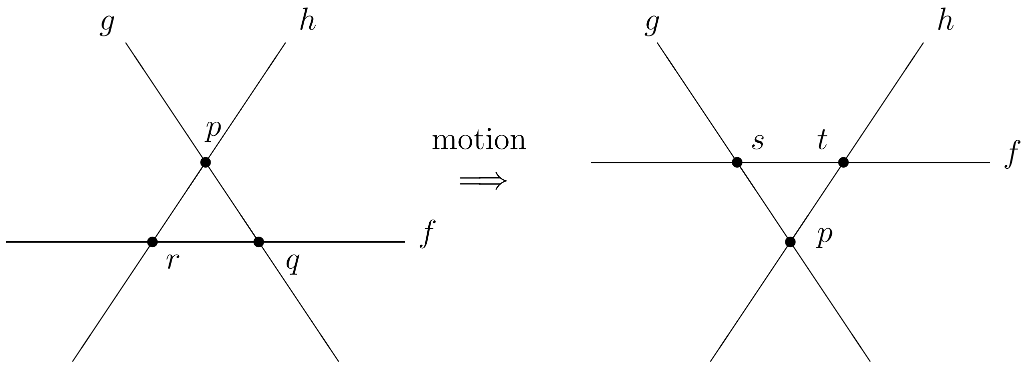

FIGURE 1.3

By §6 in [2] a $Y - \Delta$ transformation in Γ corresponds to a motion in $\mathcal{M}(\Gamma)$.

2 Combining Two Critical Graphs with Interiorization

Let Γ_1 be a critical c.c.p. graph with n boundary nodes. Let Γ_2 be a critical c.c.p. graph with m boundary nodes. Suppose we choose k successive boundary nodes of Γ_1 and k successive boundary nodes of Γ_2. By a renumbering of the boundary nodes of Γ_1 and Γ_2, we may assume that the boundary nodes of Γ_1 are $\{p_1, p_2, ..., p_k, ..., p_n\}$ ordered clockwise on the circle C_1 and the boundary nodes of Γ_2 are $\{q_1, q_2, ..., q_k, ..., q_m\}$ ordered counterclockwise on the circle C_2.

Definition 1. *The combination of* Γ_1 *and* Γ_2*, denoted* $\Gamma_1 \vee \Gamma_2$*, is formed by identifying* p_i *with* q_i *for* $i = 1, 2, ..., k$ *and then interiorizing the identified nodes.*

$\Gamma_1 \vee \Gamma_2$ is a c.c.p. graph embedded in the plane with $n+m-2k$ boundary nodes $\{p_{k+1}, p_{k+2}, ..., p_n, q_m, q_{m-1}, ..., q_{k+1}\}$ ordered clockwise on the circle C.

The z-sequence of Γ_1 is a sequence of numbers from 1 to n where each number occurs exactly twice. The z-sequence of Γ_2 is a sequence of numbers from 1 to m where each number occurs exactly twice. The first $2k$ geodesics in $\mathcal{M}(\Gamma_1)$ are distinct, and the first $2k$ geodesics in $\mathcal{M}(\Gamma_2)$ are distinct. We order $z_1 = \{1, 2, ..., 2k, P\}$ clockwise around C_1 and we order $z_2 = \{1, 2, ..., 2k, Q\}$ counterclockwise around C_2. P is a permutation of the remaining geodesic labels in z_1, and Q is a permutation of the remaining geodesic labels in z_2. Combining Γ_1 and Γ_2 joins the first $2k$ geodesics in z_1 with the first $2k$ geodesics in z_2. The z-sequence of $\Gamma_1 \vee \Gamma_2$ is $z = \{P, Q'\}$ ordered clockwise around C, where the elements of Q' are the elements of Q in reverse order.

$\Gamma_1 \vee \Gamma_2$ is pictured in Figure 2.1. The boundary nodes are circled in $\Gamma_1 \vee \Gamma_2$. The nodes that have been identified are connected with double dashed lines. The nodes $p_1 = q_1, p_2 = q_2, ..., p_k = q_k$ have been interiorized, thus they are now interior nodes and are not circled.

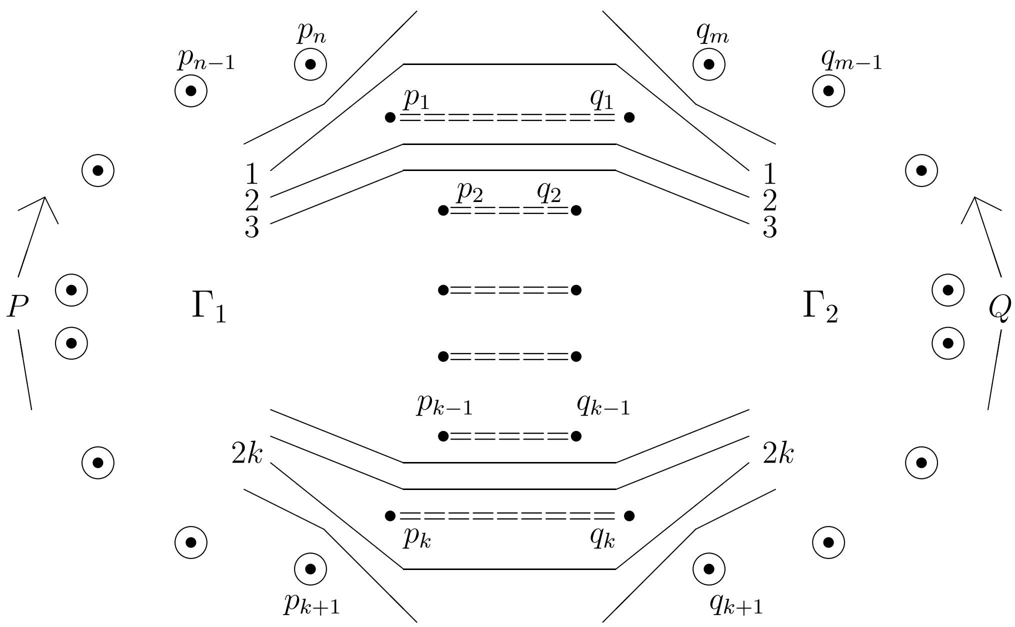

FIGURE 2.1

In $\Gamma_1 \vee \Gamma_2$, the geodesics numbered $1, 2, .., 2k$ in $\mathcal{M}(\Gamma_1)$ will be joined with the corresponding geodesics in $\mathcal{M}(\Gamma_2)$. The geodesics numbered $2k+1, 2k+2, ..., n$ in $\mathcal{M}(\Gamma_1)$ and the geodesics numbered $2k+1, 2k+2, ..., m$ in $\mathcal{M}(\Gamma_2)$ will not be joined in $\mathcal{M}(\Gamma_1 \vee \Gamma_2)$. We call the geodesics that will be joined affected geodesics and we call the geodesics that will not be joined unaffected geodesics.

Lemma 2. *Let Γ_1 and Γ_2 be critical c.c.p. graphs. $\mathcal{M}(\Gamma_1 \vee \Gamma_2)$ has no loops or self-intersections.*

Proof. Since Γ_1 and Γ_2 are both critical, there are no loops, self-intersections, or lenses in $\mathcal{M}(\Gamma_1)$ or $\mathcal{M}(\Gamma_2)$ by Corollary 6.4 in [2]. Since the geodesics numbered $1, 2, ..., 2k$ are distinct, each affected geodesic is joined exactly once. Therefore, $\mathcal{M}(\Gamma_1 \vee \Gamma_2)$ has no loops or self-intersections. □

$\mathcal{M}(\Gamma_1 \vee \Gamma_2)$ has no loops or self-intersections; however, $\mathcal{M}(\Gamma_1 \vee \Gamma_2)$ may have lenses. Finding these lenses is addressed in §4.

3 Geodesic Columns

A geodesic column is an ordered set of geodesics. Given a geodesic column A and geodesics $a, b \in A$, moving a to the position of b and moving b to the position of a is a switch of a and b. A switch forms a new geodesic column A'.

Three geodesic columns represent the affected geodesics of $\mathcal{M}(\Gamma_1)$ and $\mathcal{M}(\Gamma_2)$. The center column C will represent the affected geodesics of $\mathcal{M}(\Gamma_1)$ and $\mathcal{M}(\Gamma_2)$ in the region of the identified nodes. $C = \{1, 2, ..., 2k\}$ since the affected geodesics in this region are ordered $1, 2, ..., 2k$ in both z_1 and z_2. The left column L will represent the affected geodesics of P in reverse order. Likewise, the right column R will represent the affected geodesics of Q in reverse order.

Example 3. *An example of a combined medial graph $\mathcal{M}(\Gamma_1 \vee \Gamma_2)$ is shown in Figure 3.1a and the corresponding left, center, and right geodesic columns are shown in Figure 3.1b. Here Γ_1 has eight boundary nodes, Γ_2 has six boundary nodes, and $\Gamma_1 \vee \Gamma_2$ is formed by identifying and interiorizing three boundary* *nodes. Note that the geodesics numbered 7 and 8 in $\mathcal{M}(\Gamma_1)$ are unaffected geodesics and therefore these numbers do not appear in the columns.*

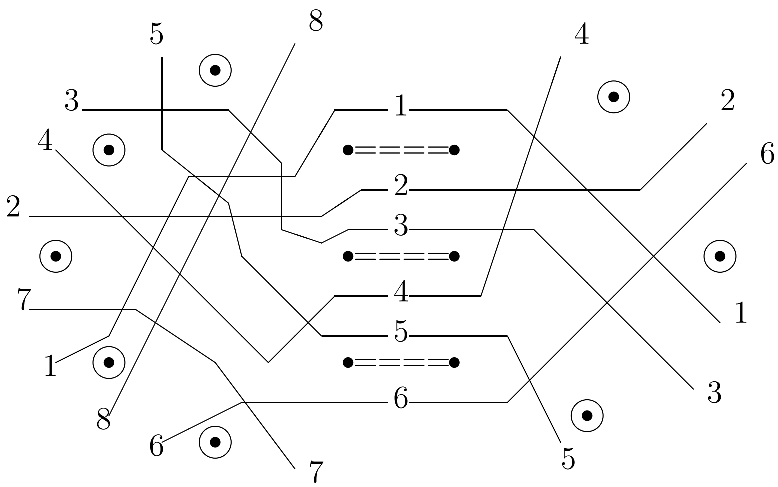

FIGURE 3.1a

<u>L</u>	<u>C</u>	<u>R</u>
5	1	4
3	2	2
4	3	6
2	4	1
1	5	3
6	6	5

FIGURE 3.1b

4 Finding Lenses

$\mathcal{M}(\Gamma_1 \vee \Gamma_2)$ has no loops or self-intersections by Lemma 2. However, the affected geodesics from $\mathcal{M}(\Gamma_1)$ and $\mathcal{M}(\Gamma_2)$ may form lenses in $\mathcal{M}(\Gamma_1 \vee \Gamma_2)$. To find these lenses, we first examine the affected geodesic pairs of $\mathcal{M}(\Gamma_1)$ and $\mathcal{M}(\Gamma_2)$ that intersect.

The set of affected geodesic pairs (a, b) that intersect in $\mathcal{M}(\Gamma_1)$ will be denoted $\mathcal{I}(\Gamma_1)$. $\mathcal{I}(\Gamma_1)$ can be determined from z_1. Since the geodesic columns L and C represent the affected geodesics of z_1, $\mathcal{I}(\Gamma_1)$ can be determined from columns L and C. Likewise, $\mathcal{I}(\Gamma_2)$ can be determined from z_2 and therefore from columns C and R. For example, if a precedes b in L and b precedes a in C then the geodesics a and b intersect in $\mathcal{M}(\Gamma_1)$, and we write $(a, b) \in \mathcal{I}(\Gamma_1)$. Using columns to determine $\mathcal{I}(\Gamma_1)$ is analogous to using numbers that interlace in a z-sequence to determine the geodesics that intersect in a medial graph, as described in §7 of [2]. However, interlacing considers all geodesics, and we need only consider affected geodesics.

The set of affected geodesic pairs (a,b) that form lenses in $\mathcal{M}(\Gamma_1 \vee \Gamma_2)$, denoted $\Psi(\Gamma_1 \vee \Gamma_2)$, is the intersection of $\mathcal{I}(\Gamma_1)$ and $\mathcal{I}(\Gamma_2)$. That is, for geodesics a and b in $\mathcal{M}(\Gamma_1 \vee \Gamma_2)$, if $(a,b) \in \mathcal{I}(\Gamma_1)$ and $(a,b) \in \mathcal{I}(\Gamma_2)$ then the geodesics a and b form a lens in $\mathcal{M}(\Gamma_1 \vee \Gamma_2)$ and we write $(a,b) \in \Psi(\Gamma_1 \vee \Gamma_2)$.

5 Removing Lenses

Definition 4. *Let C be the center geodesic column for $\mathcal{M}(\Gamma_1 \vee \Gamma_2)$ and let $(a,b) \in \Psi(\Gamma_1 \vee \Gamma_2)$. Let $P_C(a)$ denote the position of a in the center column C. Without loss of generality, we can assume $P_C(a) < P_C(b)$. Let $S_{a,b} = \{s \mid P_C(a) \leq P_C(s) \leq P_C(b)\}$. Then the gap of (a,b) in $\mathcal{M}(\Gamma_1 \vee \Gamma_2)$, written gap$(a,b)$, is defined to be the cardinality of $S_{a,b} \setminus \{a,b\}$. That is, gap$(a,b)$ is the number of geodesics between geodesics a and b in C. If gap$(a,b) = 0$, then geodesics a and b are adjacent in C. The lens formed by the geodesic pair (a,b) surrounds a lens if there exist $c, d \in S_{a,b}$ such that $(c,d) \in \Psi(\Gamma_1 \vee \Gamma_2)$ and $(c,d) \neq (a,b)$.* [1]

If geodesics a and b are adjacent in C, then the lens formed by a and b can be made empty by motions in $\mathcal{M}(\Gamma_1 \vee \Gamma_2)$. An example is pictured in Figure 5.1.

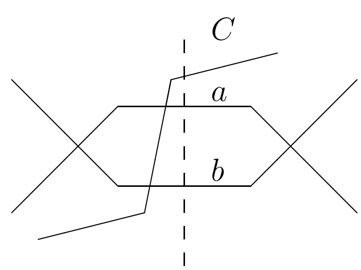

FIGURE 5.1

In both Figures 5.2a and 5.2b, the lens formed by geodesics a and b surrounds the lens formed by geodesics c and d.

[1]Note that a lens cannot surround itself since $(c,d) \neq (a,b)$; however, a lens formed by the geodesic pair (a,b) may, for example, surround a lens formed by the geodesic pair (a,c) if $c \in S_{a,b}$ and $(a,c) \in \Psi(\Gamma_1 \vee \Gamma_2)$.

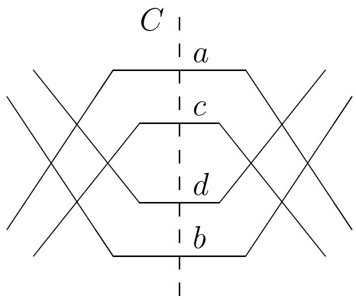

FIGURE 5.2a

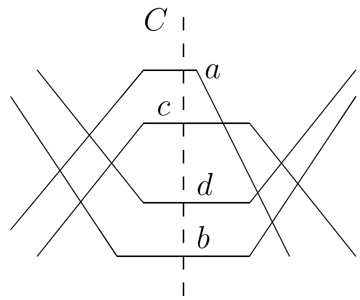

FIGURE 5.2b

Lemma 5. *Suppose* Γ_1 *and* Γ_2 *are critical c.c.p. graphs and* $\Psi(\Gamma_1 \vee \Gamma_2) \neq \emptyset$. *If* $(a, b) \in \Psi(\Gamma_1 \vee \Gamma_2)$ *has minimum gap, then the lens formed by the geodesics* a *and* b *does not surround a lens.*

Proof. Let $\mathcal{L}$ be the lens formed by the geodesics a and b in $\mathcal{M}(\Gamma_1 \vee \Gamma_2)$. Suppose $\mathcal{L}$ surrounds a lens $\mathcal{N}$. $\mathcal{N}$ is formed by some geodesic pair (a_k, b_k) $\in \Psi(\Gamma_1 \vee \Gamma_2)$ where $(a, b) \neq (a_k, b_k)$. Then $\text{gap}(a_k, b_k) \leq \text{gap}(a, b)$. This contradicts the fact that the pair of geodesics (a, b) has minimum gap. Thus $\mathcal{L}$ does not surround a lens. $\square$

Let Γ_1 and Γ_2 be critical c.c.p. graphs. Let $\Gamma_3 = \Gamma_1 \vee \Gamma_2$, and let $\mathcal{L}$ be an empty lens in $\mathcal{M}(\Gamma_3)$. Let (a, b) be the geodesic pair that forms $\mathcal{L}$. $\mathcal{L}$ results from two edges in series or in parallel in the graph Γ_3. If there is an interior node of Γ_3 inside $\mathcal{L}$, $\mathcal{L}$ results from two edges in series. If there is not an interior node of Γ_3 inside $\mathcal{L}$, $\mathcal{L}$ results from two edges in parallel. See Figure 5.3a. Let e_1 and e_2 be the two edges in series or parallel. Replacing the series or parallel combination of e_1 and e_2 in Γ_3 with a single edge e will remove the empty lens $\mathcal{L}$ from $\mathcal{M}(\Gamma_3)$. See Figure 5.3b.

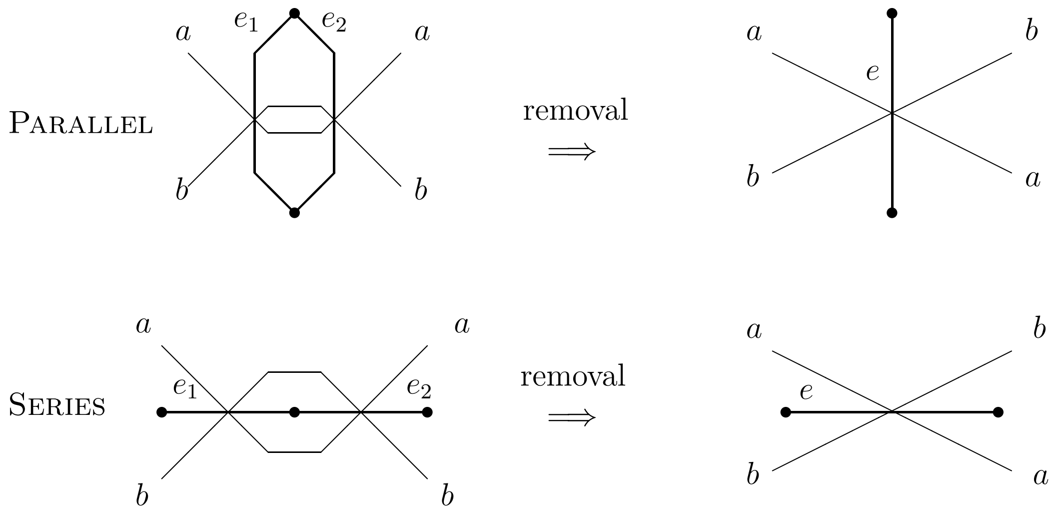

FIGURE 5.3a FIGURE 5.3b

When the empty lens $\mathcal{L}$ is removed from $\mathcal{M}(\Gamma_3)$, one geodesic intersection of a and b is eliminated and the geodesics a and b are <u>redirected</u> as shown in Figure 5.4.

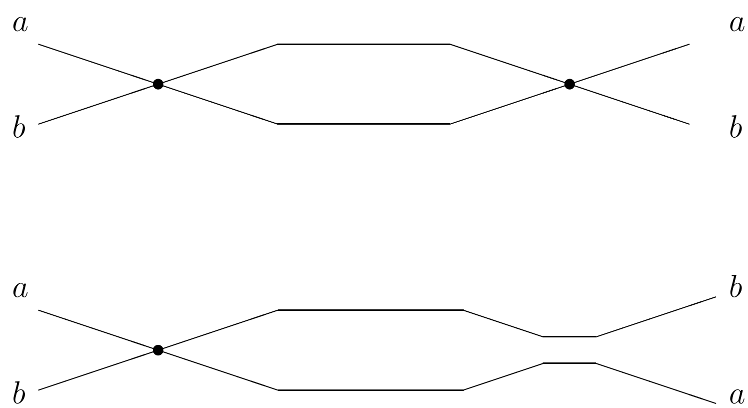

FIGURE 5.4

Replacing the edges e_1 and e_2 with the single edge e in Γ_3 creates a new graph Γ_3'. The z-sequence of the original graph Γ_3 is $z = \{P, Q'\}$ as defined in §2. Thus, the z-sequence of Γ_3' is $z' = \{P, Q''\}$ where Q'' is the permutation of Q' formed by redirecting the geodesics a and b in the lens removal process.

We will systematically remove lenses in $\mathcal{M}(\Gamma_1 \vee \Gamma_2)$ to form a critical graph Γ. Each step in the lens removal process retains the set of connections, therefore $\pi(\Gamma) = \pi(\Gamma_1 \vee \Gamma_2)$.

Lemma 6. *Two critical c.c.p. graphs have the same set of connections if their medial graphs are equivalent under motions.*

Proof. Suppose Γ_1 and Γ_2 are critical c.c.p. graphs where $\mathcal{M}(\Gamma_1)$ and $\mathcal{M}(\Gamma_2)$ are equivalent under motions. By Lemma 6.1 in [2], Γ_1 and Γ_2 are $Y - \Delta$ equivalent. Then by Theorem 1.3 in [2], $\pi(\Gamma_1) = \pi(\Gamma_2)$. □

Lemma 7. *Given two c.c.p. graphs Γ_1 and Γ_2, let $\Gamma_3 = \Gamma_1 \vee \Gamma_2$. Let $\mathcal{L}$ be an empty lens in $\mathcal{M}(\Gamma_3)$ formed by a geodesic pair (a, b). The removal of $\mathcal{L}$ from $\mathcal{M}(\Gamma_3)$ removes an edge in Γ_3 forming Γ_3'. In this case, $(a, b) \notin \Psi(\Gamma_3')$ and $\pi(\Gamma_3') = \pi(\Gamma_3)$.*

Proof. The geodesics a and b intersect twice in $\mathcal{M}(\Gamma_3)$. The lens removal process results in the elimination of a geodesic intersection and the redirection of the geodesics. Thus $(a, b) \notin \Psi\ (\Gamma_3')$.

The empty lens $\mathcal{L}$ in $\mathcal{M}(\Gamma_3)$ results from either two edges in series or two edges in parallel in Γ_3. The lens removal process replaces these two edges with a single edge and forms Γ_3'. In either case, the set of connections is retained; therefore $\pi(\Gamma_3') = \pi(\Gamma_3)$. □

Lemma 8. *Let Γ_3 be a c.c.p. graph and $(a, b) \in \Psi(\Gamma_3)$. If gap$(a, b)$ is the minimum gap, then there is a finite sequence of $Y - \Delta$ transformations in Γ_3 that forms Γ_3' such that $(a, b) \in \Psi(\Gamma_3')$ and gap$(a, b) = 0$ in $\mathcal{M}(\Gamma_3')$.*

Proof. Suppose Γ_3 is a c.c.p. graph and the geodesic pair (a, b) forms a lens $\mathcal{L}$ in $\mathcal{M}(\Gamma_3)$. Suppose also that gap(a, b) is the minimum gap. Let $S_{a,b} \backslash \{a, b\} = \{s_1, s_2, ..., s_\delta\}$, that is, $s_1, s_2, ..., s_\delta$ are the geodesics strictly between a and b in column C of $\mathcal{M}(\Gamma_3)$.

By Lemma 5, $\mathcal{L}$ does not surround a lens. Therefore, each s_i intersects a exactly once and b exactly once, and for $i, j = 1, 2, .., \delta$, $i \neq j$, s_i intersects

s_j at most once. From the proof of Lemma 6.2 in [2], a finite sequence of motions will remove from $\mathcal{L}$ every geodesic $s_1, s_2, ..., s_\delta$. Since motions in $\mathcal{M}(\Gamma_3)$ correspond to $Y-\Delta$ transformations in Γ_3 by §6 in [2], this produces Γ_3' that is $Y-\Delta$ equivalent to Γ_3. The geodesics a and b are adjacent in the center column C in $\mathcal{M}(\Gamma_3')$, thus gap(a,b) in $\mathcal{M}(\Gamma_3')$ is 0. □

Algorithm 9. *Given two critical c.c.p. graphs Γ_1 and Γ_2, let $\Gamma_3 = \Gamma_1 \vee \Gamma_2$. The algorithm consists of the following steps:*

1. *Determine the lenses in $\mathcal{M}(\Gamma_3)$.*
2. *Consider a lens $\mathcal{L}$ in $\mathcal{M}(\Gamma_3)$ formed by the geodesic pair (a,b) where gap(a,b) is minimum.*
3. *Empty $\mathcal{L}$ to form $\mathcal{M}(\Gamma_3')$.*
4. *Remove $\mathcal{L}$ to form $\mathcal{M}(\Gamma_3'')$.*
5. *Repeat steps 1-4 until $\mathcal{M}(\Gamma_3^{(n)})$ has no lenses.*

Theorem 10. *Suppose Γ_1 and Γ_2 are two critical c.c.p. graphs. Let $\Gamma_3 = \Gamma_1 \vee \Gamma_2$. Let Γ be a critical c.c.p. graph produced by Algorithm 9. Then $\pi(\Gamma)$ $= \pi(\Gamma_3)$.*

Proof. By Lemma 2, $\mathcal{M}(\Gamma_3)$ has no loops or self-intersections. We consider two cases since $\mathcal{M}(\Gamma_3)$ may or may not have lenses.

Case I. Suppose $\Psi(\Gamma_3) = \emptyset$. Then $\mathcal{M}(\Gamma_3)$ has no lenses. In this case, $\Gamma = \Gamma_3$. •

Case II. Suppose $\Psi(\Gamma_3) \neq \emptyset$. Consider $(a,b) \in \Psi(\Gamma_3)$ such that gap(a,b) is the minimum gap. Then by Lemma 5, the lens $\mathcal{L}$ formed by geodesics a and b does not surround a lens.

By Lemma 8 there exists a finite sequence of $Y-\Delta$ transformations in Γ_3 which produces Γ_3' where $(a,b) \in \Psi(\Gamma_3')$ and gap$(a,b) = 0$ in $\mathcal{M}(\Gamma_3')$. Since Γ_3' is $Y-\Delta$ equivalent to Γ_3, by Lemma 5.1 in [2] $\pi(\Gamma_3') = \pi(\Gamma_3)$.

Since gap$(a,b) = 0$ in $\mathcal{M}(\Gamma_3')$, $\mathcal{L}$ can be made empty by motions in $\mathcal{M}(\Gamma_3')$. By Lemma 7 the removal of $\mathcal{L}$ from $\mathcal{M}(\Gamma_3')$ produces $\mathcal{M}(\Gamma_3'')$ where (a,b) $\notin \Psi(\Gamma_3'')$ and $\pi(\Gamma_3'') = \pi(\Gamma_3')$.

The removal of $\mathcal{L}$ from $\mathcal{M}(\Gamma_3')$ may simultaneously create or eliminate other lenses in $\mathcal{M}(\Gamma_3')$. Therefore the sets of intersections, $\mathcal{I}(\Gamma_1'')$ and $\mathcal{I}(\Gamma_2'')$, and the set of lenses, $\Psi(\Gamma_3'')$ must be evaluated with the same process that was used to evaluate $\mathcal{I}(\Gamma_1)$, $\mathcal{I}(\Gamma_2)$, and $\Psi(\Gamma_3)$.

The removal of an empty lens from $\mathcal{M}(\Gamma_3)$ removes an edge from Γ_3. There is a finite number of edges in Γ_3. The number of edges in Γ_3 places an upper bound on the number of lenses that can occur. Therefore, a finite number of lens removals must produce a medial graph $\mathcal{M}(\Gamma_3^{(n)})$ with no lenses, that is, $\Psi(\Gamma_3^{(n)}) = \emptyset$. Since the set of connections is retained in each step of the lens removal process, $\pi(\Gamma_3^{(n)}) = \pi(\Gamma_3)$. In this case, $\Gamma = \Gamma_3^{(n)}$. •

Now, $\mathcal{M}(\Gamma)$ has no loops, self-intersections, or lenses, and by Proposition 13.1 in [2], Γ is a critical c.c.p. graph. Every step used to form Γ retains the set of connections, thus $\pi(\Gamma) = \pi(\Gamma_3) = \pi\ (\Gamma_1 \vee \Gamma_2)$. □

Corollary 11. *Let Γ_1 and Γ_2 be critical c.c.p. graphs. Let Γ and Γ' be two critical c.c.p. graphs resulting from an application of Algorithm 9. Then Γ and Γ' are $Y - \Delta$ equivalent.*

Proof. Γ and Γ' result from Algorithm 9, therefore, by Theorem 10, $\pi(\Gamma) = \pi(\Gamma_1 \vee \Gamma_2)$ and $\pi(\Gamma') = \pi(\Gamma_1 \vee \Gamma_2)$. Thus, $\pi(\Gamma) = \pi(\Gamma')$ and by Theorem 1.3 in [2], Γ and Γ' are $Y - \Delta$ equivalent. □

Applying Algorithm 9 to Example 3 results in a critical graph with eight boundary nodes and a new z-sequence $z = \{7, 6, 8, 1, 7, 2, 4, 3, 5, 8, 1, 2, 6, 4, 3, 5\}$. Additional examples are listed in §7.

6 Combining Two Graphs Without Interiorization

In §2 we defined the combination of two critical c.c.p. graphs with interiorization, $\Gamma_1 \vee \Gamma_2$. Here we define the combination of two critical c.c.p. graphs without interiorization. The following definition directly corresponds to Definition 1.

Let Γ_1 be a critical c.c.p. graph with n boundary nodes. Let Γ_2 be a critical c.c.p. graph with m boundary nodes. Suppose we choose k successive boundary nodes of Γ_1 and k successive boundary nodes of Γ_2. By a renumbering of the boundary nodes of Γ_1 and Γ_2, we may assume that the boundary nodes of Γ_1 are $\{p_1, p_2, ..., p_k, ..., p_n\}$ ordered clockwise on the circle C_1 and the boundary nodes of Γ_2 are $\{q_1, q_2, ..., q_k, ..., q_m\}$ ordered counterclockwise on the circle C_2.

Definition 12. *The* <u>*combination without interiorization*</u> *of* Γ_1 *and* Γ_2 *is formed by identifying* p_i *with* q_i *for* $i = 1, 2, ...k$ *and interiorizing the inner* $k-2$ *nodes:* $p_2 = q_2, p_3 = q_3, ..., p_{k-1} = q_{k-1}$.

The combination of Γ_1 and Γ_2 without interiorization is a c.c.p. graph embedded in the plane with $n+m-(2k-2)$ boundary nodes $\{p_k = q_k, p_{k+1}, ..., p_n, p_1 = q_1, q_m, q_{m-1}, ..., q_{k+1}\}$ ordered clockwise around the circle C. See Figure 6.1.

The z-sequence of Γ_1 is a sequence of numbers from 1 to n where each number occurs exactly twice. The z-sequence of Γ_2 is a sequence of numbers from 1 to m where each number occurs exactly twice. The first $2k-2$ geodesics in $\mathcal{M}(\Gamma_1)$ are distinct, and the first $2k-2$ geodesics in $\mathcal{M}(\Gamma_2)$ are distinct. We order $z_1 = \{1, 2, ..., 2k-2, P\}$ clockwise around C_1 and we order $z_2 = \{1, 2, ..., 2k-2, Q\}$ counterclockwise around C_2. P is a permutation of the remaining geodesic labels in z_1, and Q is a permutation of the remaining geodesic labels in z_2. Combining Γ_1 and Γ_2 without interiorization joins the first $2k-2$ geodesics in z_1 with the first $2k-2$ geodesics in z_2. The z-sequence of the combined graph is $z = \{P, Q'\}$ ordered clockwise around C, where the elements of Q' are the elements of Q in reverse order.

Note that in Figure 6.1 the geodesics above node $p_1 = q_1$ and below node $p_k = q_k$ are not joined. These geodesics are not joined because neither $p_1 = q_1$ nor $p_k = q_k$ is interiorized.

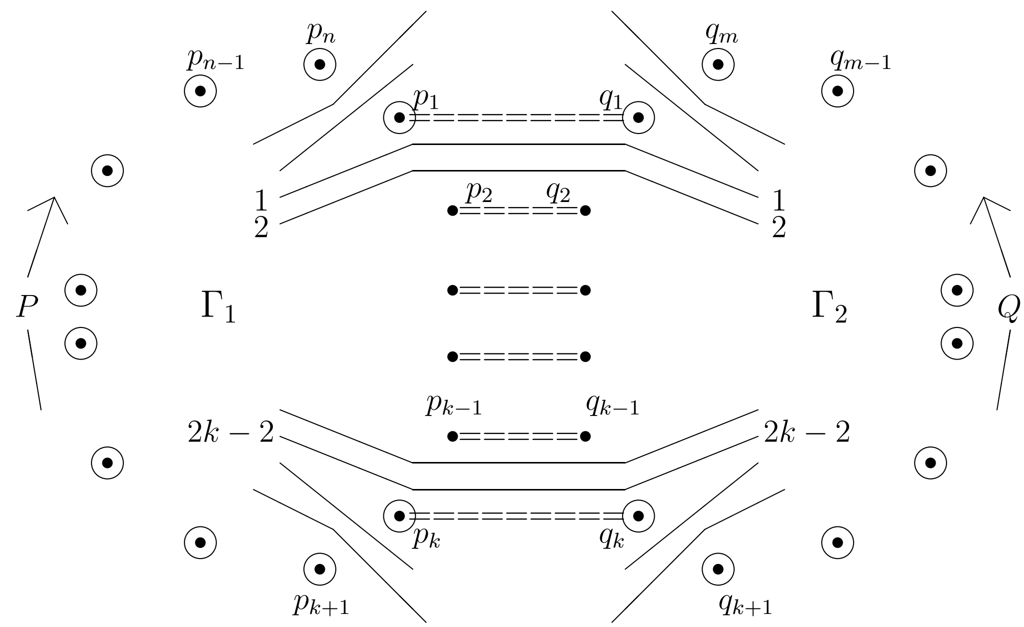

FIGURE 6.1

An algorithm very similar to Algorithm 9 will reduce the combination of Γ_1 and Γ_2 without interiorization to a critical graph with the same set of connections.

7 An Implementation of Algorithm 9

Let Γ_1 and Γ_2 be critical c.c.p. graphs. Algorithm 9 produces a critical c.c.p. graph Γ from $\Gamma_1 \vee \Gamma_2$. By Theorem 10, $\pi(\Gamma) = \pi(\Gamma_1 \vee \Gamma_2)$. We have written a computer program that implements Algorithm 9. The program takes the z-sequence of Γ_1 and the z-sequence of Γ_2 and finds z, the z-sequence of Γ. By Corollary 11 and Theorem 7.2 in [2], z is unique.

Example 13. *We give the program the following input:*

1. Γ_1 *has six boundary nodes ordered clockwise on* C_1*,*
2. Γ_2 *has eight boundary nodes ordered counterclockwise on* C_2*,*

3. *Identify and interiorize three boundary nodes from each graph to form* $\Gamma_1 \vee \Gamma_2$*,*

4. $z_1 = \{1, 2, 3, 4, 5, 6, 2, 5, 1, 4, 3, 6\}$ *ordered clockwise around* C_1*, and*

5. $z_2 = \{1, 2, 3, 4, 5, 6, 7, 8, 4, 7, 2, 5, 1, 3, 6, 8\}$ *ordered counterclockwise around* C_2*.*

In Example 13, the program produces the z-sequence of a critical c.c.p. graph Γ such that $\pi(\Gamma) = \pi(\Gamma_1 \vee \Gamma_2)$. In this case, Γ has eight boundary nodes ordered clockwise around C and $z = \{2, 5, 1, 4, 3, 6, 8, 2, 5, 1, 4, 3, 7, 6, 8, 7\}$ ordered clockwise around C.

Example 14. *Another example that is particularly interesting is:*

1. Γ_1 *has* n *boundary nodes ordered clockwise on* C_1*,*

2. Γ_2 *has* n *boundary nodes ordered counterclockwise on* C_2*,*

3. *Identify and interiorize* $n/2$ *boundary nodes from each graph to form* $\Gamma_1 \vee \Gamma_2$*,*

4. $z_1 = \{1, 2, ..., n, 1, 2, ..., n\}$ *ordered clockwise around* C_1*, and*

5. $z_2 = \{1, 2, ..., n, 1, 2, ..., n\}$ *ordered counterclockwise around* C_2*.*

In Example 14, the program produces the z-sequence of a critical c.c.p. graph Γ such that $\pi(\Gamma) = \pi(\Gamma_1 \vee \Gamma_2)$. In this case, Γ has n boundary nodes ordered clockwise around C and $z = \{1, 2, ..., n, 1, 2, ..., n\}$ ordered clockwise around C. The new graph Γ has the same z-sequence as both Γ_1 and Γ_2, thus Γ is $Y - \Delta$ equivalent to Γ_1 and Γ_2!

8 Circular Planar Networks

Thus far, we have considered combining two critical c.c.p. graphs Γ_1 and Γ_2. We have shown that $\Gamma_1 \vee \Gamma_2$ can be reduced to a graph Γ that is critical and have given an algorithm to produce Γ.

A c.c.p. network $\Omega = (\Gamma, \gamma)$ is a c.c.p. graph Γ with an associated conductivity function γ. γ maps each edge in Γ to a positive real number, γ: $E \to \mathbf{R}^+$. The number $\gamma(e_{ij})$ is called the conductivity of the edge e_{ij}.

A c.c.p. network $\Omega = (\Gamma, \gamma)$ is a critical network if its associated graph Γ is a critical graph. Let $\Omega_1 = (\Gamma_1, \gamma_1)$ and $\Omega_2 = (\Gamma_2, \gamma_2)$ be critical networks. The combination of Ω_1 and Ω_2 is $\Omega_1 \vee \Omega_2 = (\Gamma_1 \vee \Gamma_2, \gamma_1 \vee \gamma_2)$. $\Gamma_1 \vee \Gamma_2$ is formed by Definition 1. Since every edge from Γ_1 and Γ_2 is an edge in $\Gamma_1 \vee \Gamma_2$, $\gamma_1 \vee \gamma_2$ is defined as γ_1 on edges from Γ_1 and γ_2 on edges from Γ_2.

For convenience, let $\Omega_3 = \Omega_1 \vee \Omega_2, \Gamma_3 = \Gamma_1 \vee \Gamma_2$, and $\gamma_3 = \gamma_1 \vee \gamma_2$.

Algorithm 9 reduces Γ_3 to a critical graph Γ such that $\pi(\Gamma) = \pi(\Gamma_3)$. In this algorithm there are three ways to change Γ_3:

1. by a $Y - \Delta$ transformation,
2. by replacing two edges in series with a single edge, and
3. by replacing two edges in parallel with a single edge.

When we perform a $Y - \Delta$ transformation in Γ_3, Lemma 5.3 in [2] defines a conductivity function γ_3' on the graph after the transformation such that $\Lambda(\Gamma_3', \gamma_3') = \Lambda(\Gamma_3, \gamma_3)$.

When we replace two edges e_1 and e_2 in series with a new edge e, the new conductivity function is defined as

$$\gamma_3'(e) = \left(\frac{1}{\gamma_3(e_1)} + \frac{1}{\gamma_3(e_2)} \right)^{-1}.$$

When we replace two edges e_1 and e_2 in parallel with a new edge e, the new conductivity function is defined as

$$\gamma_3'(e) = \gamma_3(e_1) + \gamma_3(e_2).$$

Defining the conductivities in this way ensures that $\Lambda(\Gamma_3', \gamma_3') = \Lambda(\Gamma_3, \gamma_3)$, that is, the networks $\Omega_3' =(\Gamma_3', \gamma_3')$ and $\Omega_3 =(\Gamma_3, \gamma_3)$ have the same electrical response.

Since we reduce Γ_3 to Γ and γ_3 to γ keeping $\Lambda(\Gamma, \gamma) = \Lambda(\Gamma_3, \gamma_3)$, we have formed $\Omega = (\Gamma, \gamma)$ such that $\Lambda(\Omega) = \Lambda(\Omega_3)$.

References

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[3] Kevin Rosema, *Operations on Networks of Discrete and Generalized Conductors*.