A Study of Resistor Networks Formed of Symmetric Layered Trees

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Abstract

In this paper the electrical properties of resistor networks in the form of symmetric layered trees and the matrices governing their responses to the application of boundary voltages will be discussed. An algorithm for finding the set of edges and nodes for a given boundary response will be given for this class of matrices.

1 Introduction

This paper considers resistor networks formed from layered trees. These trees are formed by the following process:

1) Begin with a central node with a_1 edges emanating from it. This central node will be called *level 0* and the set of nodes which are connected to it by a single edge will be called *level 1*. Each of these a_1 edges connecting *level 0* to *level 1* has the same conductance, γ_1.

2) From the outer set of nodes, of which there are a_1, there will in turn be an additional layer of a_2 additional edges emanating from each *level 1* node, giving all the nodes *level 1* $a_2 + 1$ edges incident with it.

3) The new set of outer edges will be called *level 2*. Every node in *level 2* will be connected to a node in *level 1* by a single edge. There will be a_2 *level 2* nodes connected to each *level 1* node by a single edge.

4) For a tree with more layers of resistors about the central node this process can be continued by adding a_w new edges to each level $w - 1$ node

to create another layer, where $w = 3, 4, 5, ..., n-1, n$ and n is the desired number of la yers of resistors about the central node.

Definition 1 *The class of trees generated by the above process will be called SLT's and they will be further identified by subscripts. The most general tree of this form will be termed* $SLT_{a_1,a_2,a_3,...,a_{n-1},a_n}$*. The* a_i*'s represent the number of branchings at each successive layer. The central node, represented by the leftmost value, has* a_1 *edges emanating from it. Every node of level u* $(u \geq 1)$ *will have* $a_{u+1}+1$ *edges incident with it.*

For example an *SLT* with four incident edges at the node on *level 0*, 7 incident edges at each node on *level 1*, and 3 incident edges at each node on *level 2* will be called $SLT_{4,6,2}$. Note that this is not $SLT_{4,7,3}$ which has 8 incident edges at each node on *level 1* and 4 incident edges at each node on *level 2*.

Let p be a *level 1* node and q be a *level 2* node in a graph of an *SLT*, Γ, and pq be the edge connecting nodes p and q. Then the conductance on the edge pq, $\gamma(pq)$, depends only on p. So $\gamma(pq) = \gamma(p)$.

Definition 2 *Let p be a level n node and q be a level n+1 node n a graph of an SLT,* Γ*, and pq be the edge connecting nodes p and q. Then,* $\gamma(pq) = \gamma(p)$ *for every edge* $pq \in \Gamma$*.*

Definition 3 *An* **n-level** *graph is a graph of the form* $SLT_{a_1,a_2,a_3,...,a_{n-1},a_n}$*.*

Definition 4 *A* **boundary node** *is any node in the* n^{th} *level of an n-level graph.*

Definition 5 *An* **interior node** *is any node that is not a boundary node in an SLT.*

Definition 6 *Let p be an* $n-1^{th}$ *level node in an n-level graph. The set of nodes* $q_1, q_2, ..., q_{a_n}$ *which are the only* $n^{th}-$*level nodes connected by an edge to node p is to be termed a* **boundary cluster**.

Theorem 1 *Every* $SLT_{a_1,a_2,a_3,...,a_{n-1},a_n}$ *has exactly*

$$\mathcal{R} = \prod_{i=1}^{n} a_i$$

boundary nodes.

Proof: On *level 0* there are a_1 edges coming out of it. At *level 1* these a_1 edges each branch into a_2 new edges. This gives us $a_1 * a_2$ total boundary nodes at this stage. Each of these $a_1 * a_2$ edges branches into a_3 new edges at *level 2*. This gives $a_1 * a_2 * a_3$ edges at this stage. Continuing this process at the p^{th} level you will have $a_1 * a_2 * a_3 * ... * a_{p-1} * a_p$ boundary nodes. So for the graph of n levels there will be

$$a_1 * a_2 * a_3 * ... * a_{n-1} * a_n = \prod_{i=1}^{n} a_i$$

boundary nodes. //

Theorem 2 *Every* $SLT_{a_1,a_2,a_3,...,a_{n-1},a_n}$ *has exactly*

$$\mathcal{J} = \prod_{i=1}^{n-1} a_i$$

boundary clusters.

Proof: On *level 0* there are a_1 edges coming out of it. At *level 1* these a_1 edges each branch into a_2 new edges giving $a_1 * a_2$ *level 2* nodes. In a 3 level *SLT* the boundary clusters would be attached to these $a_1 * a_2$ nodes. For an n level *SLT* there are $a_1 * a_2 * a_3 * ... * a_{n-1} = \prod_{i=1}^{n-1} a_i$ nodes at *level n-1*. Attatched to each of these n-1 nodes is a boundary cluster. //

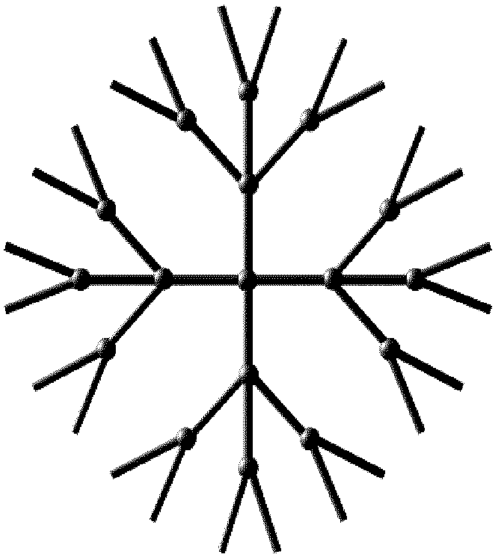

Figure 1: This is $SLT_{4,3,2}$. It has 24 boundary nodes, 4 levels [levels $0-4$], and 17 interior nodes which are pictured as circles.

Definition 7 *The boundary nodes will be ordered and individually referenced as δ_i.*

Where $i = 1, 2, 3, ..., \mathcal{R}$.

Definition 8 *The sets of boundary clusters will be numbered $C_1, C_2, C_3, ..., C_{\prod_{i=1}^{n-1} a_i}$. Each set C_i will contain the ordered boundary nodes $\delta_{(i-1)\cdot a_n + j}$ where $i = 1, 2, 3, ..., \prod_{i=1}^{n-1} a_i$ and $j = 1, 2, 3, ..., a_n$.*

2 The Response Matrix

The Response matrix Λ can found both experimentally or by computation from the known conductances and the set of connections of a network. The form of Λ can also be found for *SLT's* with only the knowledge of the set of connections, without directly computing the boundary currents due to the application of voltages at the boundary. This result is an application of a result given by [1] which is shown here.

Result 1

$$det\Lambda\left(\mathcal{P};\mathcal{Q}\right)\cdot detK\left(I,I\right)=\left(-1\right)^{k}\sum_{r\in S_k} sgn\left(\tau\right)\left\{\begin{array}{l}\sum_{\alpha\in P}\prod_{e\in E_\alpha}\gamma\left(e\right)\cdot detK\left(J_\alpha;J_\alpha\right)\\ \tau_\alpha=\tau\end{array}\right\}$$

This is a way to compute the values in the Λ matrix. It can also be used to find those values in the Λ matrix for which the calculations are the same. ¿From this a pattern based on the number of boundary nodes in the boundary clusters appears.

$$\Lambda=\begin{bmatrix}
A & a & a & b & b & b & c & c & c & d & d & d\\
a & A & a & b & b & b & c & c & c & d & d & d\\
a & a & A & b & b & b & c & c & c & d & d & d\\
e & e & e & F & f & f & g & g & g & h & h & h\\
e & e & e & f & F & f & g & g & g & h & h & h\\
e & e & e & f & f & F & g & g & g & h & h & h\\
i & i & i & j & j & j & K & k & k & l & l & l\\
i & i & i & j & j & j & k & K & k & l & l & l\\
i & i & i & j & j & j & k & k & K & l & l & l\\
m & m & m & n & n & n & o & o & o & P & p & p\\
m & m & m & n & n & n & o & o & o & p & P & p\\
m & m & m & n & n & n & o & o & o & p & p & P
\end{bmatrix}$$

Figure 2:

3 Recovering The Networks Connections

The connections of an *SLT* can be recovered by finding the Λ matrix corresponding to the n-1 level network which underlies an n level network.

Theorem 3 *Let p be a node in level n-1 and q_i be a level node connected to p by a single edge. The Λ matrix of the n-1 level network underlying an n level SLT can be recovered by inducing a potential of 0 volts at all nodes in level n-1 except p, and inducing a 1 volt potential at p.*

Proof: The above mentioned induced potentials are the process of generating the Λ matrix of the the n-1 level network by definition of the Λ matrix.

For every node r in level n-1 there are exactly $a_n + 1$ incident edges. Of these $a_n + 1$ incident edges a_n of them connect to the a_n nodes in the boundary cluster associated with r. The other edge, w, connected to r connects to a level n-2 node. The current through w is equal to the sum of the currents through the a_n edges connecting r to the boundary cluster by Kirchhoff's Law. So if the proper voltages are induced on the set of nodes r, then the sum of the currents through the a_n edges connecting r to the boundary cluster is equal to the current through node r and is the entry in the Λ matrix c orresponding to that node. This process can be done over all the nodes giving the new Λ matrix. //

Knowing the process needed to derive the reduced Λ matrix, we now need a way to induce a 1 volt potential and a 0 volt potential at a level n-1 node. This requires knowledge of the conductance of the edges incident with the boundary nodes.

Theorem 4 *The conductance of an edge pq, where $q \in C_i$ and p is the level n-1 node associated with C_i, is equal to the sum of $\lambda_{i\cdot a_n+1,i\cdot a_n+1}$ and the negative of $\lambda_{i\cdot a_n+2,i\cdot a_n+1}$.*

$$\gamma\left(pq\right) = \lambda_{i\cdot a_n+1,i\cdot a_n+1} - \lambda_{i\cdot a_n+2,i\cdot a_n+1}$$

Proof: Let δ_j and δ_{j+1} be members of the same boundary cluster C_i. Apply a 1 volt potential at δ_j, a -1 volt potential at δ_{j+1}, and a 0 volt potential at all the other boundary nodes. This will induce a voltage of 0 volts at node p. These induced voltages give the current through δ_{j+1}.

$$I\left(\delta_{j+1}\right) = \lambda_{i\cdot a_n+2,i\cdot a_n+2} - \lambda_{i\cdot a_n+2,i\cdot a_n+1}$$

Where $j+1 = i \cdot a_n + 2$. While $I = \gamma \cdot \delta V$ and $\delta V = 1$. So:

$$\lambda_{i\cdot a_n+2,i\cdot a_n+2} - \lambda_{i\cdot a_n+2,i\cdot a_n+1} = \gamma$$

It is also true that $\lambda_{i\cdot a_n+2,i\cdot a_n+2} = \lambda_{i\cdot a_n+1,i\cdot a_n+1}$ by Result 1. //

Knowing the cunductance of an edge pq connection a boundary and an interior node we can then induce a voltage of 1 volt at the interior node p.

Theorem 5 *Let p be a level n-1 node and $q = \delta_k$, where $\delta_k \in C_j$, be a level n node in an n-level graph. To induce a 1 volt potential at p, a voltage of $V_{gamma(pq)}$ must be applied to node q. Where :*

$$V_{\gamma(pq)} = \frac{-\lambda_{i\cdot a_n+2,i\cdot a_n+2} + \lambda_{i\cdot a_n+2,i\cdot a_n+1}}{\lambda_{i\cdot a_n+2,i\cdot a_n+1}}$$

Proof: This can be done by direct computation. //

This can then be used to acquire the Λ matrix for the n-1 level network underlying an n level network. This is done by applying 1 volt to each of the $\mathcal{J}$ n-1 level nodes and 0 volts to the rest, and then taking the sum of the currents leaving each boundary cluster. This will give a matrix of the form seen in Figure 3 for the matrix in Figure 2 and a matrix which has undergone this process will be called a reduced matrix.

$$\Lambda_2 = \begin{bmatrix} \frac{a-A}{a}\cdot(A+2\cdot a) & \frac{f-F}{f}\cdot(3\cdot b) & \frac{k-K}{k}\cdot(3\cdot c) & \frac{p-P}{p}\cdot(3\cdot d) \\ \frac{a-A}{a}\cdot(3\cdot e) & \frac{f-F}{f}\cdot(F+2\cdot f) & \frac{k-K}{k}\cdot(3\cdot g) & \frac{p-P}{p}\cdot(3\cdot h) \\ \frac{a-A}{a}\cdot(3\cdot i) & \frac{f-F}{f}\cdot(3\cdot j) & \frac{k-K}{k}\cdot(K+2\cdot k) & \frac{p-P}{p}\cdot(3\cdot l) \\ \frac{a-A}{a}\cdot(3\cdot m) & \frac{f-F}{f}\cdot(e\cdot n) & \frac{k-K}{k}\cdot(3\cdot o) & \frac{p-P}{p}\cdot(P+2\cdot p) \end{bmatrix}$$

Figure 3: This is n-1 level reduced form of the Λ matrix in Figure 2.

If the reduced matrix in Figure 3 is from $SLT_{2,2,3}$ then

$$\begin{gathered} \Lambda_{2_{1,1}} = \Lambda_{2_{2,2}} \\ \Lambda_{2_{1,2}} = \Lambda_{2_{2,1}} \\ \Lambda_{2_{1,3}} = \Lambda_{2_{1,4}} = \Lambda_{2_{2,3}} = \Lambda_{2_{2,4}} = \Lambda_{2_{3,1}} = \Lambda_{2_{3,2}} = \Lambda_{2_{4,1}} = \Lambda_{2_{4,2}} \\ \Lambda_{2_{3,3}} = \Lambda_{2_{4,4}} \\ \Lambda_{2_{3,4}} = \Lambda_{2_{4,3}} \end{gathered}$$

These equalities allow us to place algebraic restrictions on the possible values for the variables which make up the Λ matrix as in Figure 3. These results of solving these equalities, or their analogues for another network, for specific variables allows the creation of a network with imposed conditions of specific boundary currents due to the application of 1 volt to a boundary node. The information in this paper forms a basis on which to build a more general method of creating resistor networks in the form of trees with specific desired properties.

4 Reference

These papers provide the basis of what this work is based upon along with other information pertaining to resistor networks.

[1] E. B. Curtis,D. Ingerman, and J. A. Morrow, *Circular Planar Graphs And Resistor Networks*, preprint (1994) .

[2] E. B. Curtis and J. A. Morrow, *The Dirichlet to Neumann Map for a Resistor Network*, SIAM J. of Applied Math, 51 (1991), pp. 1011-1029.

[3] E. B. Curtis, E. Mooers, and J. A. Morrow, *Finding the Conductors in Circular Networks from Boundary Measurements*, To appear in 1994.