A Study of Resistor Networks Formed of Symmetric Layered Trees

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Abstract

In this paper the electrical properties of resistor networks in the form of symmetric layered trees and the matrices governing their responses to the application of boundary voltages will be discussed. An algorithm for finding the set of edges and nodes for a given boundary response will be given for this class of matrices.

1 Introduction

This paper considers resistor networks formed from layered trees. These trees are formed by the following process:

1 ) Begin with a central node with $a_1$ edges emanating from it. This central node will be called level 0 and the set of nodes which are connected to it by a single edge will be called level 1. Each of these $a_1$ edges connecting level 0 to level 1 has the same conductance, $\gamma_1$.

2 ) From the outer set of nodes, of which there are $a_1$, there will in turn be an additional layer of $a_2$ additional edges emanating from each level 1 node, giving all the nodes level 1 $a_2 + 1$ edges incident with it.

3 ) The new set of outer edges will be called level 2. Every node in level 2 will be connected to a node in level 1 by a single edge. There will be $a_2$ level 2 nodes connected to each level 1 node by a single edge.

4 ) For a tree with more layers of resistors about the central node this process can be continued by adding $a_w$ new edges to each level $w - 1$ node
to create another layer, where \( w = 3, 4, 5, \ldots, n - 1, n \) and \( n \) is the desired number of layers of resistors about the central node.

\section*{Definition 1} The class of trees generated by the above process will be called SLT’s and they will be further identified by subscripts. The most general tree of this form will be termed \( SLT_{a_1, a_2, a_3, \ldots, a_{n-1}, a_n} \). The \( a_i \)'s represent the number of branchings at each successive layer. The central node, represented by the leftmost value, has \( a_1 \) edges emanating from it. Every node of level \( u \) \((u \geq 1)\) will have \( a_{u+1} + 1 \) edges incident with it.

For example an SLT with four incident edges at the node on level 0, 7 incident edges at each node on level 1, and 3 incident edges at each node on level 2 will be called \( SLT_{4,6,2} \). Note that this is not \( SLT_{4,7,3} \) which has 8 incident edges at each node on level 1 and 4 incident edges at each node on level 2.

Let \( p \) be a level \( 1 \) node and \( q \) be a level \( 2 \) node in a graph of an SLT, \( \Gamma \), and \( pq \) be the edge connecting nodes \( p \) and \( q \). Then the conductance on the edge \( pq \), \( \gamma(pq) \), depends only on \( p \). So \( \gamma(pq) = \gamma(p) \).

\section*{Definition 2} Let \( p \) be a level \( n \) node and \( q \) be a level \( n+1 \) node in a graph of an SLT, \( \Gamma \), and \( pq \) be the edge connecting nodes \( p \) and \( q \). Then, \( \gamma(pq) = \gamma(p) \) for every edge \( pq \in \Gamma \).

\section*{Definition 3} An \( n \)-level graph is a graph of the form \( SLT_{a_1, a_2, a_3, \ldots, a_{n-1}, a_n} \).

\section*{Definition 4} A boundary node is any node in the \( n^{th} \) level of an \( n \)-level graph.

\section*{Definition 5} An interior node is any node that is not a boundary node in an SLT.

\section*{Definition 6} Let \( p \) be an \( n-1^{th} \) level node in an \( n \)-level graph. The set of nodes \( q_1, q_2, \ldots, q_{a_n} \) which are the only \( n^{th} \) – level nodes connected by an edge to node \( p \) is to be termed a boundary cluster.

\section*{Theorem 1} Every \( SLT_{a_1, a_2, a_3, \ldots, a_{n-1}, a_n} \) has exactly

\[ \mathcal{R} = \prod_{i=1}^{n} a_i \]

boundary nodes.
Proof: On level 0 there are $a_1$ edges coming out of it. At level 1 these $a_1$ edges each branch into $a_2$ new edges. This gives us $a_1 \times a_2$ total boundary nodes at this stage. Each of these $a_1 \times a_2$ edges branches into $a_3$ new edges at level 2. This gives $a_1 \times a_2 \times a_3$ edges at this stage. Continuing this process at the $p^{th}$ level you will have $a_1 \times a_2 \times a_3 \times \ldots \times a_{p-1} \times a_p$ boundary nodes. So for the graph of $n$ levels there will be

$$a_1 \times a_2 \times a_3 \times \ldots \times a_{n-1} \times a_n = \prod_{i=1}^{n} a_i$$

boundary nodes. //

Theorem 2 Every $\text{SLT}_{a_1, a_2, a_3, \ldots, a_{n-1}, a_n}$ has exactly

$$\mathcal{J} = \prod_{i=1}^{n-1} a_i$$

boundary clusters.

Proof: On level 0 there are $a_1$ edges coming out of it. At level 1 these $a_1$ edges each branch into $a_2$ new edges giving $a_1 \times a_2$ level 2 nodes. In a 3 level $\text{SLT}$ the boundary clusters would be attached to these $a_1 \times a_2$ nodes. For an $n$ level $\text{SLT}$ there are $a_1 \times a_2 \times a_3 \times \ldots \times a_{n-1} = \prod_{i=1}^{n-1} a_i$ nodes at level $n-1$. Attached to each of these $n-1$ nodes is a boundary cluster. //
Figure 1: This is $SLT_{3,2}$. It has 24 boundary nodes, 4 levels [levels 0−4], and 17 interior nodes which are pictured as circles.

**Definition 7** The boundary nodes will be ordered and individually referenced as $\delta_i$.
Where $i = 1, 2, 3, ..., \mathcal{R}$.

**Definition 8** The sets of boundary clusters will be numbered $C_1, C_2, C_3, ..., C_{\prod_{i=1}^{n-1} a_i}$.
Each set $C_i$ will contain the ordered boundary nodes $\delta_{(i-1)a_n+j}$ where $i = 1, 2, 3, ..., \prod_{i=1}^{n-1} a_i$ and $j = 1, 2, 3, ..., a_n$.

**2 The Response Matrix**

The Response matrix $\Lambda$ can found both experimentally or by computation from the known conductances and the set of connections of a network. The form of $\Lambda$ can also be found for $SLT$’s with only the knowledge of the set of connections, without directly computing the boundary currents due to the application of voltages at the boundary. This result is an application of a result given by [1] which is shown here.
Result 1

\[ \text{det} \Lambda (\mathcal{P}; \mathcal{Q}) \cdot \text{det} K (I, I) = (-1)^k \sum_{r \in S_k} \text{sgn} (\tau) \left\{ \sum_{\alpha \in P} \Pi_{\varepsilon \in E_\alpha} \gamma (e) \cdot \text{det} K (J_\alpha; J_\alpha) \right\} \]

This is a way to compute the values in the \( \Lambda \) matrix. It can also be used to find those values in the \( \Lambda \) matrix for which the calculations are the same. From this a pattern based on the number of boundary nodes in the boundary clusters appears.

\[ \Lambda = \begin{bmatrix}
A & a & a & b & b & b & c & c & c & d & d & d \\
A & a & a & b & b & b & c & c & c & d & d & d \\
a & a & a & A & b & b & b & c & c & c & d & d & d \\
e & e & e & F & F & F & f & f & g & g & g & h & h & h \\
e & e & e & f & F & F & f & f & g & g & g & h & h & h \\
e & e & e & f & f & F & g & g & g & h & h & h & h & h \\
i & i & i & j & j & j & j & K & k & k & k & l & l & l \\
i & i & i & j & j & j & k & K & k & k & l & l & l & l \\
i & i & i & j & j & j & k & k & K & k & k & l & l & l \\
m & m & m & n & n & n & o & o & o & P & p & p & p & p \\
m & m & m & n & n & n & o & o & o & p & P & P & p & p \\
m & m & m & n & n & n & o & o & o & p & p & P & P & p & p \\
\end{bmatrix} \]

Figure 2:

3 Recovering The Networks Connections

The connections of an SLT can be recovered by finding the \( \Lambda \) matrix corresponding to the n-1 level network which underlies an n level network.

**Theorem 3** Let \( p \) be a node in level n-1 and \( q_i \) be a level node connected to \( p \) by a single edge. The \( \Lambda \) matrix of the n-1 level network underlying an n level SLT can be recovered by inducing a potential of 0 volts at all nodes in level n-1 except \( p \), and inducing a 1 volt potential at \( p \).

**Proof:** The above mentioned induced potentials are the process of generating the \( \Lambda \) matrix of the the n-1 level network by definition of the \( \Lambda \) matrix.
For every node $r$ in level $n-1$ there are exactly $a_{n+1}$ incident edges. Of these $a_{n+1}$ incident edges $a_n$ of them connect to the $a_n$ nodes in the boundary cluster associated with $r$. The other edge, $w$, connected to $r$ connects to a level $n-2$ node. The current through $w$ is equal to the sum of the currents through the $a_n$ edges connecting $r$ to the boundary cluster by Kirchhoff’s Law. So if the proper voltages are induced on the set of nodes $r$, then the sum of the currents through the $a_n$ edges connecting $r$ to the boundary cluster is equal to the current through node $r$ and is the entry in the $\Lambda$ matrix corresponding to that node. This process can be done over all the nodes giving the new $\Lambda$ matrix.

Knowing the process needed to derive the reduced $\Lambda$ matrix, we now need a way to induce a 1 volt potential and a 0 volt potential at a level $n-1$ node. This requires knowledge of the conductance of the edges incident with the boundary nodes.

**Theorem 4** The conductance of an edge $pq$, where $q \in C_i$ and $p$ is the level $n$ node associated with $C_i$, is equal to the sum of $\lambda_{i,a_n+1,i,a_n+1}$ and the negative of $\lambda_{i,a_n+2,i,a_n+1}$.

$$\gamma(pq) = \lambda_{i,a_n+1,i,a_n+1} - \lambda_{i,a_n+2,i,a_n+1}$$

**Proof:** Let $\delta_j$ and $\delta_{j+1}$ be members of the same boundary cluster $C_i$. Apply a 1 volt potential at $\delta_j$, a -1 volt potential at $\delta_{j+1}$, and a 0 volt potential at all the other boundary nodes. This will induce a voltage of 0 volts at node $p$. These induced voltages give the current through $\delta_{j+1}$.

$$I(\delta_{j+1}) = \lambda_{i,a_n+2,i,a_n+2} - \lambda_{i,a_n+2,i,a_n+1}$$

Where $j + 1 = i \cdot a_n + 2$. While $I = \gamma \cdot \delta V$ and $\delta V = 1$. So:

$$\lambda_{i,a_n+2,i,a_n+2} - \lambda_{i,a_n+2,i,a_n+1} = \gamma$$

It is also true that $\lambda_{i,a_n+2,i,a_n+2} = \lambda_{i,a_n+1,i,a_n+1}$ by Result 1. //

Knowing the conductance of an edge $pq$ connection a boundary and an interior node we can then induce a voltage of 1 volt at the interior node $p$.

**Theorem 5** Let $p$ be a level $n$ node and $q = \delta_k$, where $\delta_k \in C_j$, be a level $n$ node in an $n$-level graph. To induce a 1 volt potential at $p$, a voltage of $V_{\gamma\text{ma}(pq)}$ must be applied to node $q$. Where:

$$V_{\gamma}(pq) = \frac{-\lambda_{i,a_n+2,i,a_n+2} + \lambda_{i,a_n+2,i,a_n+1}}{\lambda_{i,a_n+2,i,a_n+1}}$$

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Proof: This can be done by direct computation. //

This can then be used to acquire the $\Lambda$ matrix for the $n-1$ level network underlying an $n$ level network. This is done by applying 1 volt to each of the $J$ $n-1$ level nodes and 0 volts to the rest, and then taking the sum of the currents leaving each boundary cluster. This will give a matrix of the form seen in Figure 3 for the matrix in Figure 2 and a matrix which has undergone this process will be called a reduced matrix.

$$
\Lambda_2 = \begin{bmatrix}
\frac{a-A}{a} \cdot (A + 2 \cdot a) & \frac{1}{f} \cdot (3 \cdot b) & \frac{k-K}{k} \cdot (3 \cdot c) & \frac{p-P}{p} \cdot (3 \cdot d) \\
\frac{a-A}{a} \cdot (3 \cdot e) & \frac{1}{f} \cdot (F + 2 \cdot f) & \frac{k-K}{k} \cdot (3 \cdot g) & \frac{p-P}{p} \cdot (3 \cdot h) \\
\frac{a-A}{a} \cdot (3 \cdot i) & \frac{1}{f} \cdot (3 \cdot j) & \frac{k-K}{k} \cdot (K + 2 \cdot k) & \frac{p-P}{p} \cdot (3 \cdot l) \\
\frac{a-A}{a} \cdot (3 \cdot m) & \frac{1}{f} \cdot (e \cdot n) & \frac{k-K}{k} \cdot (3 \cdot o) & \frac{p-P}{p} \cdot (P + 2 \cdot p)
\end{bmatrix}
$$

Figure 3: This is $n-1$ level reduced form of the $\Lambda$ matrix in Figure 2.

If the reduced matrix in Figure 3 is from $SLT_{2,3}$ then

$$
\Lambda_{2,1,1} = \Lambda_{2,2,2} \\
\Lambda_{2,1,2} = \Lambda_{2,2,1} \\
\Lambda_{2,1,3} = \Lambda_{2,4,1} = \Lambda_{2,3,3} = \Lambda_{2,4,2} = \Lambda_{2,3,1} = \Lambda_{2,4,1} = \Lambda_{2,1} = \Lambda_{2,2} \\
\Lambda_{2,3,3} = \Lambda_{2,4,4} \\
\Lambda_{2,4,3} = \Lambda_{2,4,4}
$$

These equalities allow us to place algebraic restrictions on the possible values for the variables which make up the $\Lambda$ matrix as in Figure 3. These results of solving these equalities, or their analogues for another network, for specific variables allows the creation of a network with imposed conditions of specific boundary currents due to the application of 1 volt to a boundary node. The information in this paper forms a basis on which to build a more general method of creating resistor networks in the form of trees with specific desired properties.

4 Reference

These papers provide the basis of what this work is based upon along with other information pertaining to resistor networks.


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