Ψ Inverse Conductivity Problem

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1 Introduction

We deal with electrical circuits considered as *networks* where *edges* are conductors. There are two types of the nodes of the network: *interior* and *boundary*. We can impose and measure *voltages* and *currents* on the boundary. The first *Kirchoff's law* is true within the interior: all current, that comes into an interior node, will leave it.

There is a linear map Λ from boundary voltages to boundary currents. Original problem is to recover conductances of the network if we know the structure of the network (its shape) and Λ - how much the voltages influence the currents. I change the problem considering a Ψ-matrix instead Λ.

2 Preliminaries

A network is represented by its *Kirchoff matrix*. It is a square matrix $N \times N$: $K = \{k_{i,j}\}$ (where N is the number of nodes in the network). Nodes are numbered so that boundary nodes come first. Any non-diagonal entry $k_{i,j}$ equals minus *conductance* of the edge between nodes i and j (if there is no edge between them then the conductance is 0). And diagonal entries are sums of the conductances of the edges incident to the node.

The Kirchoff matrix (*K-matrix*) is symmetric, entries of any row or column sum up to 0. All entries of the matrix are negative or zero, if there is no edge. Only diagonal entries are positive. This matrix contains all information about the network.

The edges of the network are divided in groups so that the first group consists of edges from boundary to boundary, the second: between the boundary and the interior and the last one consists of edges which connect two interior nodes. K-matrix is also divided in blocks by the same rule:

$$K = \begin{array}{|c|c|} \hline K' & B^T \\ \hline B & A \\ \hline \end{array}$$

The sizes of blocks are determined by the number of boundary nodes: N_b and interior nodes: N_i.

The standard inverse problem considers the recovering K-matrix from Λ-matrix. There is a formula for Λ:

$$\Lambda = K^{'} - B^T A^{-1} B.$$

I consider not Λ but Ψ:

$$\Psi = B^T A^{-1} B$$
$$\Lambda = K^{'} - \Psi.$$

We consider only connected networks. In the article I consider networks with no boundary to boundary edges so $K^{'}$ is a diagonal matrix.

For detailed description see [1], [2].

3 Basic theorem

The following theorem 3.4 will be important:

Lemma 3.1 Suppose S and T are two m×n matrices such that:

$$S^T S = T^T T, \tag{1}$$

and there some $\alpha_1, \alpha_2, .., \alpha_n$ (where at least one α is non-zero) such that:

$$\sum_{i=1}^{n} \alpha_i s_i = 0, \tag{2}$$

where s_i is a i^{th} column of S, then,

$$\sum_{i=1}^{n} \alpha_i t_i = 0,$$

where t_i is a i^{th} column of T.

Proof We will consider matrices S and T as sets of column vectors. Take inner product of the sum of (2) and any s_j:

$$(\sum_{i=1}^{n} \alpha_i s_i) s_j = 0.$$

But, using (1):

$$(\sum_{i=1}^{n} \alpha_i s_i) s_j = \sum_{i=1}^{n} \alpha_i (s_i, s_j) = \sum_{i=1}^{n} \alpha_i (t_i, t_j) = (\sum_{i=1}^{n} \alpha_i t_i) t_j.$$

Thus we have,

$$(\sum_{i=1}^{n} \alpha_i t_i) t_j = 0.$$

Thereafter:

$$(\sum_{i=1}^{n} \alpha_i t_i)^2 = \sum_{j=1}^{n} (\sum_{i=1}^{n} \alpha_i t_i) \alpha_j t_j =$$

$$= \sum_{j=1}^{n} ((\sum_{i=1}^{n} \alpha_i t_i) t_j) \alpha_j = \sum_{j=1}^{n} (0) \alpha_j = 0.$$

And so,

$$\sum_{i=1}^{n} \alpha_i t_i = 0.$$

Corollary 3.2 Linear independence is equivalent in both matrices.

Corollary 3.3

$$\mathrm{Ifforas_j:}\ \ s_j = \sum_{i=1}^{n} \alpha_i s_i, \quad \text{then} \ \ \sum_{i=1}^{n} \alpha_i t_i.$$

Theorem 3.4 Suppose S and T are two m×n matrices such that:

$$S^T S = T^T T, \tag{3}$$

than there is a m×m orthogonal matrix R ($R^T R = I$) such that $T = RS$.

Proof. I will consider matrices S and T as sets of column vectors. Then R will map columns of S into T's.

Reorder the columns of matrix S so that some first columns form a linear independent set (call it sub-basis) while all the other are linear combinations of them. Exchange the columns of T so that the correspondence between columns of S and T will be kept. By 3.2 T will have the same structure as S.

Determine R as a operator that maps one sub-basis into the other. It is possible because the vectors are linear independent. Then the image of R on any other column of S is a linear combination of images of the sub-basis (which are sub-basis in T) where the coefficients are determined from the formula for the vector:

$$R(s_j) = R(\sum_{i=1}^{n} \alpha_i s_i) = \sum_{i=1}^{n} \alpha_i R(s_i) = \sum_{i=1}^{n} \alpha_i t_i = t_j$$

Thus R is a linear map that takes columns of S into T. To finish constructing of R we just complete both sub-bases to bases by orthonormal bases of the orthogonal complements to the sub-bases.

Consider R: it maps a basis into another preserving inner products of any pair of basis vectors (if both are from S it is clear because S and T satisfy (1), if any of two is from the complement, the product will be 0). Then inner product of any two vectors will be preserved:

$$(R(s^{'}), R(s^{"})) = (R(\sum_{i=1}^{l} \alpha_i s_i), R(\sum_{j=1}^{l} \beta_j s_j)) =$$

$$= (\sum_{i=1}^{l} \alpha_i R(s_i), \sum_{j=1}^{l} \beta_j R(s_j)) = (\sum_{i=1}^{l} \alpha_i t_i, \sum_{j=1}^{l} \beta_j t_j) =$$

$$= \sum_{i=1}^{l} \sum_{j=1}^{l} \alpha_i \beta_j (t_i, t_j) = \sum_{i=1}^{l} \sum_{j=1}^{l} \alpha_i \beta_j (s_i, s_j) =$$

$$= (\sum_{i=1}^{l} \alpha_i s_i, \sum_{j=1}^{l} \beta_j s_j) = (s^{'}, s^{"}).$$

Then it will be easy to prove that R is an orthogonal matrix. Apply R to a vector with one 1 and all the other entries 0. The result will be a column of R. But the norm of the original vector was 1, then the norm of any column of R is 1. Now apply R over a vector with 2 ones. The result will be the sum of two columns of R and the norm of the image will be 2. But

$$2 = (r^{'} + r^{"})^2 = (r^{'})^2 + (r^{"})^2 + (r^{'}, r^{"}) = 1 + 1 + (r^{'}, r^{"}).$$

Thus all columns of R are orthogonal terms. And so

$$R^T R = I.$$

Statement 3.5 If we can form a basis from the columns of S then there is only one R.

4 Networks for particular Ψ

Theorem 4.1 If a matrix equation

$$\Psi = B^T A^{-1} B, \tag{4}$$

where the sizes of matrices are known and A is square positive definite, has more than one solution then they will be connected by a linear map E:

$$B_2 = EB_1 \tag{5}$$

$$A_2 = EA_1E^T. \tag{6}$$

Proof I apply Cholesky decomposition to A-matrices, i.e., I will consider each A matrix as a product:

$$A = C^TC,$$

where C is the uniquely determined upper-triangular matrix with positive entries on the diagonal. Thus,

$$B_1^T A_1^{-1} B_1 = \Psi = B_2^T A_2^{-1} B_2$$

$$B_1^T C_1^{-1} C_1^{-1^T} B_1 = B_2^T C_2^{-1} C_2^{-1^T} B_2$$

$$(C_1^{-1^T} B_1)^T (C_1^{-1^t} B_1) = (C_2^{-1^T} B_2)^T (C_2^{-1^t} B_2).$$

Thereafter by Theorem 3.4 there is an orthogonal R such that,

$$(C_2^{-1^T} B_2) = R(C_1^{-1^T} B_1).$$

Now denoting,

$$E = C_2^T R C_1^{-1^T}, \tag{7}$$

we have,

$$B_2 = EB_1.$$

Then I want to show that E will satisfy (6):

$$E_T A_2^{-1} E = (C_1^{-1} R_T C_2) A_2^{-1} (C_2^T R C_1^{-1^T}) =$$

$$= (C_1^{-1} R_T)(C_2 C_2^{-1} C_2^{-1^T} C_2^T)(R C_1^{-1^T}) =$$

$$= (C_1^{-1} R_T)(R C_1^{-1^T}) = C_1^{-1}(R_T R) C_1^{-1^T} = C_1^{-1} C_1^{-1^T} = A_1^{-1},$$

so we have,

$$E_T A_2^{-1} E = A_1^{-1},$$

and then,

$$A_2 = EA_1E^T.$$

Theorem 4.2 If there is linear map E such that,

$$B_2 = EB_1$$

and

$$A_2 = EA_1E^T,$$

and A_1, B_1 is a solution of the equation (4):

$$B^TA^{-1}B = \Psi,$$

then A_2, B_2 will be a solution of that equation too.

Proof Just compute:

$$B_2^TA_2^{-1}B_2 = (B_1^TE^T)A_2^{-1}(EB_1) = B_1^T(E^TA_2^{-1}E)B_1 =$$
$$B_1^TA_1^{-1}B_1 = \Psi.$$

Theorem 4.3 If A_1 and A_2 are positive definite, then any F such that,

$$A_2 = FA_1F^T, \tag{8}$$

can be decomposed into,

$$F = C_2^TPC_1^{-1^T},$$

where P is an orthogonal matrix and C_1 and C_2 are triangular matrices of Cholesky decomposition of A_1 and A_2.

Proof Using Cholesky decomposition:

$$\begin{aligned} A_2 &= FA_1F^T \\ C_2^TC_2 &= F(C_1^TC_1)F^T \\ C_2^TC_2 &= (C_1F^T)^T(C_1F^T), \end{aligned}$$

and by 3.4,

$$\begin{aligned} C_2 &= P(C_1F^T) \\ C_1^{-1}P^{-1}C_2 &= F^T \\ F &= C_2^TP^{-1^T}C_1^{-1^T} \\ F &= C_2^TPC_1^{-1^T} \end{aligned}$$

5 Networks of fixed shape

In this section I assume that the network has fixed shape. The only variables are conductivities. I will need some new definitions:

Definition 5.1 Sh_B - is the set of matrices $B = \{b_{ij}\}$, $N_i \times N_b$, whose entry b_{ij} is either negative if there is an edge between nodes i and j or equal to 0 if not.

Definition 5.2 Sh_A - is the set of positive definite matrices $A = \{a_{ij}\}$, $N_i \times N_i$, whose non-diagonal entry a_{ij} is either negative if there is an edge between nodes i and j or equal to 0 if not. A has to be symmetric with positive diagonal entries.

Definition 5.3 Sh_K - is the set of K-matrices such that B $\in Sh_B$, A $\in Sh_A$ and for any row of K, corresponding to an interior node, the sum of entries is 0.

$$K = \begin{array}{|c|c|cl} \hline K' & B^T & & \\ \hline B & A & \Big| & \text{Sums of entries in these rows are 0.} \\ \hline \end{array}$$

Definition 5.4 U is the intersection of two sets of linear maps:

$$U_1 = \{D \ \text{ such that there is a } \ B \in Sh_B : DB \in Sh_B\}$$

$$and$$

$$U_2 = \{D \ \text{ such that there is a } \ A \in Sh_A : DAD^T \in Sh_A\}.$$

Any D $\in U$ is non-singular because DAD^T is non-singular, if A and B are a solution of (4) then images of them will also satisfy that equation by 4.2. Any D $\in U$ takes some K into another preserving zeros:

$$D(K) = K^{'}$$

$$\mathrm{D} \begin{array}{|c|c|} \hline K^{'} & B^T \\ \hline B & A \\ \hline \end{array} = \begin{array}{|c|c|} \hline K^{'} & (DB)^T \\ \hline DB & DAD^T \\ \hline \end{array}$$

For any D_1 and $D_2 \in U$ their product $D_2 D_1 \in U$.

Definition 5.5 $U^{'}$ is the subset of U, consisting of D which not only preserve zeros in A and B but also map a K $\in Sh_K$ into another element of Sh_K.

For any D_1 and $D_2 \in U^{'}$ their product $D_2 D_1 \in U^{'}$.

Definition 5.6 I_Z is the set of diagonal matrices I_z with positive diagonal entries (z is the vector of diagonal entries of I_z). And I is the matrix of identity.

It is easy to see that U is never empty: it always include I_Z, Because any I_z does not influence upon the shape of matrices A and B: it keeps signs of all entries.

Definition 5.7 Vector **1** is the vector whose all entries are 1.

Theorem 5.8 For any D $\in U$ there is the only z such that $I_z D \in U^{'}$. I will call this I_z - the correction to D.

Proof For any $I_z, I_z D \in U$. The only thing we need to look at are sums in the rows. Consider the matrices A and B which D keeps within the sets Sh_B and Sh_A. Now write down the equation of 0 sums of entries in a row:

$$
\begin{gathered}
(I_z DB)\mathbf{1} + (I_z DAD^T I_z)\mathbf{1} = 0 \\
I_z(DB\mathbf{1} + DAD^T z) = 0 \\
DB\mathbf{1} + DAD^T z = 0 \\
DAD^T z = -DB\mathbf{1} \\
z = -(DAD^T)^{-1} DB\mathbf{1} \\
z = (DAD^T)^{-1}(-DB)\mathbf{1} \\
z = (A^{'})^{-1}(-B^{'}\mathbf{1}).
\end{gathered}
$$

We have got the formula for z. We must check that all entries of z are positive. $B^{'} \in Sh_B$, so all entries of B are non-positive. Then $(-B^{'}\mathbf{1})$ is a vector with positive entries. All entries of $(A^{'})^{-1}$ are positive too, because $A^{'} \in Sh_A$. Thus z consists of positive entries.

Corollary 5.9 D $\in U^{'}$ iff the correction is the identity.

Corollary 5.10 $I_Z \cap U^{'} = I$.

Proof It is clear that $I_{z^{-1}}$ will be a correction for I_z, because $I_{z^{-1}} I_z = I$. So the correction is not identity unless $I_z = I$.

Corollary 5.11 If $U \neq I_Z$ then $U^{'}$ includes more then only I.

Proof We know that $I_Z \in U$. Then there is a D $\in U$ such that D $\notin I_Z$. Then there is a corrected map $I_z D$ whose correction will be the identity, and so $I_z D \in U'$. And $I_z D$ will not be the identity, because if it were the identity then D=$I_{z^{-1}}$ but D $\notin I_Z$.

Theorem 5.12 Any E $\in U'$ takes one K-matrix into another.

Proof Consider $E(K)$ where K is a K-matrix (a Kirchoff matrix of a network). $E(K)$ will have the same Ψ and the same shape. The only thing that we need to check is that entries of any of first N_b rows sum up to 0. We know that sums of entries of last N_i rows is 0 in K and $E(K)$:

$$A\mathbf{1} + B\mathbf{1} = 0$$

$$EAE^T\mathbf{1} + EB\mathbf{1} = 0.$$

We can multiply by E^{-1} the former equation because E is non-singular. Now extract $-A^{-1}B\mathbf{1}$ from both equations:

$$-A^{-1}B\mathbf{1} = \mathbf{1}$$

$$-A^{-1}B\mathbf{1} = E^T\mathbf{1}.$$

Then,

$$E^T = \mathbf{1}.$$

Look at the sum of entries of rows in $(EB)^T$:

$$(EB)^T\mathbf{1} = B^TE^T\mathbf{1} = B^T\mathbf{1}.$$

The sums have not changed. Thus the sums in K itself have not changed too. And $E(K)$ is a K-matrix.

Corollary 5.13 If $U \neq I_Z$ then a solution of the problem of recovering the network from the Ψ-matrix and the shape of the network is not unique.

Theorem 5.14 Criteria of uniqueness. If a solution of the problem of recovering the network from the Ψ-matrix and the shape of the network is not unique then $U \neq I_Z$.

Proof The non-uniqueness of the solution means that there are two different K-matrices K_1 and K_2 with the same Ψ. It means that there are two solutions of (4). Then by 4.1 there is a E that takes A_1 and B_1 into A_2 and B_2. Thus E $\in U$. But A_2 and B_2 are blocks of a K_2 and $K_2 \in Sh_K$. So E $\in U'$. Then $U \neq I_Z$ because K's are different and so E can not be identity.

6 Networks with no edges between interior nodes

These networks have diagonal A matrix. So its Cholesky decomposition will be a product of a diagonal matrix C with its transpose.

Consider equation (4):

$$\Psi = B^T A^{-1} B,$$

denoting:

$$H = CB,$$

we will have:

$$\Psi = H^T H.$$

We know that any two solutions are connected by an orthogonal matrix $R = \{r_{i,j}\}$ (see 2.1):

$$H_2 = RH_1.$$

If the only possible R were I then there would be the unique H. And E for $(5-6)$ will be by (7):

$$E = C_2^T R C_1^{T^{-1}} = C_2^T C_1^{T^{-1}} = C_2 C_1^{-1}.$$

Thus $E \in I_Z$ and it will lead to uniqueness of the whole problem.

So I want to force R to be the identity. All I know about R is that it is orthogonal and it preserves the shape of H. C is a diagonal matrix with positive entries, thereafter H has the same shape as B: $H \in Sh_B$. Thus R should preserve zeros and negative signs in H.

What should B be so that R will be the identity? I had to give up sign conditions because it was very difficult to use them. Then I considered the problem: how I should put zeros in B so that the problem of recovering the network from Ψ and shape will have unique solution not depending upon the value of conductances.

Consider R as a collection of rows: r_i. Orthogonality gives $N_i(N_i-1)/2$ non-linear equations for rows. Each zero as a value of an entry $h_{i,j}$ of H gives a linear equation for the i^{th} row of R. If R were diagonal then $R=I$ would follow from the equation of orthogonality of R: $R^TR=I$. Managing zeros in H, I want to have R diagonal.

There is a homogeneous system of linear equations for any row r_i of R consisting of equations with coefficients from the columns of H which have zeros in the i^{th} row. All coefficients of r_{ii} will be 0. So the system gives no restrictions on it. Thus the system has N_i-1 variables. The space of solutions of the system will be a subspace of a R^{N_i-1}-space. And the dimension of the subspace will be N_i-1 without the number of linear independent equations of the system.

If the system has only zero solution then the entries of the row of R will be all zeros except the diagonal entry, which is free to be any number. If it is true for systems for all rows of R then R will be diagonal, and so identity and then I will have the uniqueness of whole problem.

But I can reach the same result using fewer of zeros. I can use equations of orthogonality to eliminate dimensions in the subspaces of solutions of the systems.

Here is how to do it. If a system for a row is full (the number of unknowns is equal to the number of linear independent equations) then the row will consist of 0 except the diagonal entry. Then using orthogonality it is clear that all other rows of R must have 0 in the column of non-zero in that row. Thus the number of unknowns in the remaining systems became less. If I can continue the process this way I will have diagonal R at last.

Reorder rows in H and R in the order of consideration of the rows by the process. Then to make the process work R should have the proper amount of linear independent equations in the systems: (N_i-1) for the top row, then (N_i-2), etc. up to one for the next to the bottom row (rows are after reordering):

.	0	0	0	0
.	.	0	0	0
.	.	.	0	0
.	.	.	.	0
.	.	.	.	.

This means that for each row r_i there are enough linear independent columns of H with 0 in the i^{th} row.

This will be true if I can select a linear independent collection of columns B, so that there will be enough zeros. If it is impossible then I will have to select linear independent equations from the system for each row of R, and check the number of them.

Thus I am really interested in some kind of linear independent columns only. So after a certain moment I do not care about the rest of columns of H, i.e. I pay no attention to additional boundary nodes.

I want to find networks such that their shape will lead unique recoverability conductances of all edges. So my linear independent columns have not to lose the independence for any values of their entries. All I know about H is the positions of zeros and negativeness of all the other entries. or negative. To check linear independence I need some more things. Sum of vectors is a vector of sums of coordinates, here it will be just the same way, but an entry of the sum is 0 if it is a sum of zeros and the entry is non-zero in the other case. Two vectors are equal if they have zeros in the same places, because if so then I can choose numbers so that they will be equal and if not then there is no way to get these vectors equal.

Criteria for linear independence of a collection of vectors if we know only zero entries and we know that all other entries have the same sign:

If a vector is a sum of some vectors of the collection then it will be the unique way to express the vector as a sum of the vectors of the collection.

Example Now I want to show a non-triangular matrix B which satisfies all conditions, i.e., networks of this shape is uniquely recovering from Ψ-matrix:

.	0	0	0	.
0	.	.	0	.
.	.	0	.	.
.	0	.	.	.

Here four first columns are in use.

7 References

[1] E.Curtis and J.Morrow, *Determining the resistors in a network*.
[2] D.Ingerman, *Theory of Equivalent Networks and Some of its Applications*.