Mixed-sign Conductor Networks

Konrad Schrøder
5 March 1995

Section 1: Introduction

In this paper we consider conductor networks which possibly contain both positive and negative conductors.

A *conductor* is a two-sided object which obeys Ohm's law: if potentials V_1 and V_2 are applied respectively to each end of the conductor, then the current flowing through the conductor is given by

$$I = (V_1 - V_2)\gamma$$

where γ is an intrinsic property of the conductor, known as its *conductance*. When we speak, therefore, of "positive" and "negative" conductors, we mean conductors whose conductances are positive and negative, respectively.

A *network* of conductors is any number of conductors, joined at the ends; such joinings are known as *nodes*. If all conductances are positive, this can be represented physically by a resistor network, or more generally by a network of pipes or channels. In the networks that we consider, certain nodes form the boundary of the network, and others the interior; we restrict our ability to impose potentials or currents to the boundary only.

Because we have so restricted ourselves, Kirkhoff's current law will hold at each of the interior nodes; in the case where all conductances are positive, this implies a unique set of interior potentials due to imposed boundary potentials (the Dirichlet problem) and a unique (up to a constant) set of potentials due to imposed boundary currents (the Neumann problem). This uniqueness is essential to the construction of certain useful tools, such as the Dirichlet-to-Neumann map.

If some of the conductors are negative, the situation is more complicated: in the case of electrical networks, a negative conductor would represent a non-dissipative element, such as a generator; similarly it would represent a pump in the case of a pipe network. Uniqueness of the Dirichlet and Neumann problems is not automatic in these cases: for example, two resistors of resistance 1 and -1 placed in series will make a effectively resistance-free object; current may flow when both ends are grounded, meaning that the interior potential is something other than zero. Similarly, the same resistors placed in parallel will form a short circuit; and current might flow through the network even though both ends are insulated. Other such combinations, however, do not suffer from this problem: replace -1 by $-\frac{1}{2}$ in each case and no badly-behaved object is created.

This paper is divided into two sections, one dealing with *singular* networks, that is, networks that exhibit the bad behaviour as described above; and one dealing with planar *non-singular* networks, which appear to have some relation to non-planar networks composed entirely of positive conductors.

Section 2: Singular Networks

Section 2.1: general definitions; the Neumann problem

We begin with the Neumann problem, since it is easier to analyze than the Dirichlet problem. In particular, imposing zero current on the boundary nodes is equivalent to requiring that Kirkhoff's current law hold there; so the distinction between boundary and interior nodes is only nominal when considering solutions to the homogeneous Neumann problem.

Definition 2.1. *A network consists of a graph $\Omega = (N, E)$ along with a conductivity function γ defined on the set E of edges, such that $\gamma \neq 0$. The set N of nodes is divided into two subsets: the set N_b of boundary nodes and the set N_i of interior nodes.*

Definition 2.2. *A function f defined on the nodes of a network is said to be γ-harmonic if at every interior node i, it is true that Kirkhoff's current law holds, i.e.,*

$$\sum_j f(j)\gamma_{ij} = f(i)\sum_j \gamma_{ij}$$

Definition 2.3. *For any function f defined on the nodes of a network, the boundary current at a boundary node i is given by*

$$\sum_j (f(i) - f(j))\gamma_{ij}$$

The statement of the Neumann problem on conductor networks is this: given a vector ψ of boundary currents, is there a γ-harmonic function f defined on the nodes of Γ such that ψ gives the boundary current for f? If the conductances γ are known to be all positive, then the existence and uniqueness—up to a constant—of f are guaranteed. For conductances both positive and negative, however, neither of these things is guaranteed, since current can flow in circles.

If there are no "circles", however, for current to flow around, then we should have a well-behaved network. We look first, therefore, at networks whose graphs are trees; and then generalize to graphs that contain circuits.

Lemma 2.4. *Suppose that u is the solution to the homogeneous Neumann problem on a network Γ, and that i is a node of valence 1 in Γ. Let Γ' be the network obtained from Γ by deleting i and the edge ij,* *where j is the sole neighbor of i. Then $u|_{\Gamma'}$ is a solution to the homogeneous Neumann problem on Γ'; and $u(i) = u(j)$.*

Proof. Kirkhoff's law at i states that $u(i) = u(j)$; since this is so,

$$\sum_k (u(j) - u(k))\gamma_{jk} = \sum_{k \neq i} (u(j) - u(k))\gamma_{jk}$$

Since u satisfied Kirkhoff's law at j, then $u|_{\Gamma'}$ satisfies Kirkhoff's law at j.

Theorem 2.5. *Given a network Γ, and a subnetwork T whose underlying graph is a tree. Further suppose that there exists some node i, an endpoint of T, such that the only way to reach any node of T from a node not in T is to pass through i. Denote by S the network $\Gamma \setminus T$, where both $i \in S$ and $i \in T$. Then if u is a solution to the homogeneous Neumann problem on Γ, $u|_S$ is a solution to the homogeneous Neumann problem on S; and furthermore, for any node $j \in T$, $u(j) = u(i)$.*

Proof. Apply lemma 2.4 to the endpoints of T repeatedly until T has been entirely removed from the network, except for node i.

Corollary 2.6. *Suppose Γ is a network whose underlying graph is a tree. Then the family of solutions of the homogeneous Neumann problem is one-dimensional.*

Theorem 2.7. *Given any network Γ. The family of solutions to the homogeneous Neumann problem on Γ is at most $(n_c + 1)$-dimensional, where n_c is the number of basis circuits in the graph underlying Γ.*

Proof. Let Γ' be Γ restricted to a tree spanning its graph. By corollary 2.6, the only solutions to the homogeneous Neumann problem on Γ' are constants. For every edge $e \in (\Gamma \setminus \Gamma')$, we specify a current i_e. The current i_e must flow through the circuit closed by the addition of e to Γ'; if it does not, then some part of the current must flow across another edge e_2 not in Γ'; but then the current on e_2 would not be what we specified.

There are n_c such edges in Γ; thus the family of solutions to the homogeneous Neumann problem on Γ is at most $n_c + 1$-dimensional.

Theorem 2.8. *Suppose that Γ is a network, none of whose circuits is singular, i.e. such that the sum of the inverse of the conductances along any circuit is not 0. Then the only solutions to the homogeneous Neumann problem on Γ are constants.*

Proof. Consider a spanning tree Γ' as in the proof of theorem 2.7; and let AB be an edge of Γ not in Γ'. Applying Ohm's law repeatedly to the edges of the circuit C closed by AB gives

$$u(A) - u(B) = - \sum_{e \in C,\ e \neq AB} \frac{i_{AB}}{\gamma_e}$$

where i_{AB} is the current flow from A to B. We can also apply Ohm's law directly to AB:

$$u(A) - u(B) = \frac{i_{AB}}{\gamma_{AB}}$$

In order for these to agree, we must have

$$\sum_{e \in C} \frac{i_{AB}}{\gamma_e} = 0$$

Since this condition does not hold for any circuit C, the current i_{AB} is equal to zero. Since this is so, Kirkhoff's current law at A and B will remain unchanged if the edge AB is removed. Thus, the only solutions to the homogeneous Neumann problem are those solving the same problem on Γ'; i.e., constants.

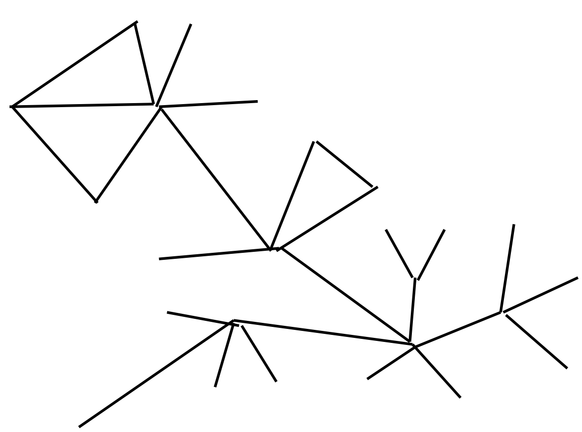
Section 2.2: the Dirichlet problem.

Now we turn to the Dirichlet problem. Here, the difference between interior and boundary nodes does not simply go away; also, we must consider the possibility of singular elements; if a network with 2 boundary nodes A and B can have a potential difference but no current flow between A and B, then it is disconnected and thus illegal; but such a network with current flowing and no potential difference between A and B is not in any way illegal.

Definition 2.9. *The* interior *of a network* $\Gamma = (N, E, \gamma)$, *denoted by* Γ° *is the network* $(N, E', \gamma|_{E'})$ *obtained from* Γ *by deleting all edges that join two boundary nodes.*

Definition 2.10. *The* skeleton *of a network* Γ *is the network obtained by deleting all boundary nodes and all edges that abut boundary nodes.*

Definition 2.11. *A network is said to be a* tree network *if the graph underlying its skeleton is a tree.*



Definition 2.12. *A tree network is said to be a linear tree network if its skeleton consists of a single chain.*

Now that our terms are well-defined, we begin our discussion of the Dirichlet problem on conductor networks. We first consider linear tree graphs, where the question of the uniqueness of solutions to the Dirichlet problem is relatively simple; we can then use these linear trees as building blocks to build up larger and more complicated graphs.

Lemma 2.13. *If u is a solution to the homogeneous Dirichlet problem on a network Γ, then u is also a solution of the same problem on Γ°.*

Proof. Suppose that u is a solution to the homogeneous Dirichlet problem on a network Γ. Since the potential at every boundary node is zero, Ohm's law applied to any boundary-to-boundary conductor gives $0 = 0\gamma$; and we can choose γ arbitrarily to satisfy this equation. In particular, we can choose $\gamma = 0$.

Lemma 2.14. *If u is a solution to the homogeneous Dirichlet problem on a network Γ, and Γ' is the network formed by removing all interior nodes of valence 1, then $u|_{\Gamma'}$ is a solution to the homogeneous Dirichlet problem on Γ'.*

Proof. Suppose i is an interior node of valence 1. Since Kirkhoff's law holds at i, no current ever flows into or out of i. We can therefore delete the edge joining i to its neighbor without disturbing the functioning of Kirkhoff's law at the neighbor.

By these lemmas, we can ignore boundary-to-boundary connections when considering whether the Dirichlet problem has a unique solution on a network; and we can ignore sections of the graph that are "almost entirely insulated". The following discussion assumes that the networks in question do not have either of these features.

Theorem 2.15. *The family of solutions to the homogeneous Dirichlet problem on a linear tree network Γ is at most one-dimensional.*

Proof. Number the interior nodes 1, 2, …, n_i, so that nodes k and $k+1$ are connected for any k strictly between 0 and n_i. Suppose that we knew the potential at node $k < n_i$ and at node $k-1$. Since the network is a linear tree network, node k is connected to at most two other interior nodes, $k-1$ and $k+1$. Knowing the potentials at all neighbors of node k but node $k+1$, Kirkhoff's law will determine the potential at node $k+1$.

Suppose that we choose a potential x at node 1. By induction, we have determined the potentials at nodes 2, 3, …, n_i. Thus the possible solutions to the Dirichlet problem on a linear tree network form a at most a one-parameter family.

Corollary 2.16. *Suppose that the potential at all boundary nodes of a linear tree network is zero, and that the potential at one end of its skeleton is known. Then the potential function u satisfying Kirkhoff's law at every point of the interior is unique, if it exists.*

Definition 2.17. *A network is said to be D-singular if any nontrivial solution to the homogeneous Dirichlet problem exists on that network.*

Theorem 2.18. *Suppose that a linear tree network is D-singular. Then any non-trivial solution u to the homogeneous Dirichlet problem on that network must have non-zero current flow at boundary nodes that are connected to nodes at the ends of the skeleton.*

Proof. Suppose that some solution had zero current flow at some boundary node B, which is connected to an interior node I, which in turn is one endpoint of the skeleton of the network. Since the network is a linear tree network, B is connected to no other interior node. Ohm's law applied to the conductor IB implies that $u(I) = u(B) = 0$. Again applying Ohm's law to any conductors that join I to boundary nodes, we find that all currents between I and boundary nodes are zero.

Now number the interior nodes $I_0, I_1, \ldots, I_n$; suppose that node I_{k-1} has potential zero, and that no current flows into I_{k-1} from another interior node (possibly excepting I_k). Applying Ohm's law to all connections joining I_{k-1} to boundary nodes, we see that no current flows into I_{k-1} from the boundary; applying Kirkhoff's law, we see further that no current flows out of I_{k-1}, and in particular no current flows between I_{k-1} and I_k. Using Ohm's law once again, we see that $u(I_k) = u(I_{k-1}) = 0$.

Thus we know by induction that the potential on every interior node is zero. But then u is not a non-trivial solution, which is a contradiction.

Theorem 2.19. *The family of solutions to the homogeneous Dirichlet problem on a tree network is at most $(n_l - 1)$-dimensional, where n_l denotes the number of leaves of the graph underlying its skeleton.*

Proof. Consider the network as a collection of subnetworks, each of which are linear tree networks. These subnetworks intersect at *branch points,* namely those nodes of the skeleton which have valence greater than two. Each subnetwork component touches either one branch point or two; if it touches only one, let the node at the other end of its skeleton be called a *leaf point.* Thus the nodes of the skeleton are divided into three classes: leaf points, which have valence 1; branch points, which have valence > 2; and other points, which are not important to our discussion. Denote by b_i the valence of the branch point i; and by n_l the number of leaf points.

Consider a branch point i, with known potential u_i and interior neighbor j whose potential is known. We can arbitrarily choose potentials at all of the other interior neighbors but one; and at the final interior neighbor Kirkhoff's law will uniquely determine the potential.

Consider the component subnetworks abutting node i, apart from that containing node j. By Corollary 2.16, the potential in these components, up to and including their other branch points (if any), is determined.

Now, we can traverse the tree recursively, considering any leaf point as the tree's root; our first choice of potential is at the root, which determines potential at the first branch point; and at every other branch point i we can make $b_i - 2$ choices of potential.

In this way we have constructed an at most $(\sum_i (b_i - 2) + 1)$-dimensional family of solutions to the homogeneous Dirichlet problem on tree networks. Since for any tree

$$\sum_i (b_i - 2) = n_l - 2,$$

this number can also be written as $n_l - 1$.

In fact, it is possible to realize this bound for any given shape of tree simply by making all components singular; in this case the potential at every branch point is 0; and we can choose $n_l - 1$ independent parameters that will result in a solution to the homogeneous Dirichlet problem.

Theorem 2.20. *The family of solutions to the homogeneous Dirichlet problem on a network Γ is at most $(n_l + n_c - 1)$-dimensional, where n_l is the number of leaf points in a tree network spanning Γ, and n_c is the number of basis circuits in the graph underlying the skeleton of Γ.*

Proof. Consider the skeleton of Γ. We can form a new network Γ' by deleting enough linear tree subnetworks from Γ that the skeleton of Γ' is a tree; we can then apply theorem 2.4 to Γ'. Now suppose that we added one of the linear trees, T, that is in Γ but not in Γ', thus forming a circuit in the new skeleton. We may be able to specify the volatge at some point in T; if we can, this gives us one more free parameter. In adding $\Gamma \setminus \Gamma'$, we have obtained at most n_c parameters; therefore, the family of solutions to the nomogeneous Dirichlet problem on Γ is at most $(n_l + n_c - 1)$-dimensional.

In some sense, this result is not as good as theorem 2.19, since for not all shapes of graph can the bound be realized; for example, networks whose skeleton consists of a single circuit consistently have only a 1-dimensional family of solutions, whereas the count in theorem 2.20 would have the dimension be $2+1-1 = 2$.

3. Relation to Planarity

We now turn to the somewhat more interesting case of connected, non-singular networks; of primary interest here is the fact that the response matrix of a connected network which is not D-singular is composed entirely of finite non-zero values.

3.1. Circular-planar graphs and circular subdeterminants

When all conductors are constrained to be positive, [3] gives a characterization of the set Ω of *circular-planar* graphs on the basis of signs of certain subdeterminants of the response matrices.

Definition 3.1. *An $n \times n$ matrix $M = (m_{ij})$will be said to be a Kirkhoff Matrix provided that the following conditions hold: (1) M is symmetric; (2) $\sum_j m_{ij} = 0$ for all i; (3) for all i, for all $j > i$, $m_{ij} < 0$. We denote by K^n the set of all $n \times n$ Kirkhoff matrices; and by K the set of all Kirkhoff matrices.*

Note that any matrix in K^n can be viewed as the response matrix for a fully-connected n-node network with strictly positive conductances.

Definition 3.2. *A Circular Planar Graph is a graph embedded in a disc D in the plane so that the boundary nodes lie on the circle C which bounds D, and the rest of Γ is in the interior of D.*

Definition 3.3. *Let Γ be a circular planar graph, with boundary nodes $v_1, \dots, v_n$. in clockwise order around C. A pair of sequences of boundary nodes $(P; Q) = (p_1, \dots, p_k; q_1, \dots, q_k)$ such that the entire sequence $(p_1, \dots, p_k, q_1, \dots, q_k)$ is in circular order is called a Circular Pair.*

Definition 3.4. *Let Γ be a circular planar graph, with boundary nodes $v_1, \dots, v_n$, and response matrix Λ; and let $(P; Q)$ be a circular pair of boundary nodes. The submatrix obtained by taking the entries of Λ that are in rows $p_1, \dots, p_k$ and in columns $q_1, \dots, q_k$ will be called a circular submatrix of Λ, and will be denoted $\Lambda(P; Q)$. $\det \Lambda(P; Q)$ will be known as a $k \times k$ circular subdeterminant of Λ.*

Theorem 3.5. (Curtis and Morrow) *Let Γ be a circular planar network of positive conductances, and let Λ be the corresponding response matrix. Then for any circular pair $(P; Q)$, $-1^{\lceil \frac{k}{2} \rceil} \det \Lambda(P; Q) > 0$, where $\lceil x \rceil$ denotes the least integer greater than x.*

Theorem 3.6. (Curtis and Morrow) *Let Λ be an $n \times n$ matrix such that $\Lambda \in K^n$, the set of Kirkhoff matrices for positive networks; and also suppose that for any circular pair $(P; Q)$, $-1^{\lceil \frac{k}{2} \rceil} \det \Lambda(P; Q) > 0$. Then there exists some positive circular planar network Γ having Λ as its response matrix.*

Proof. The proof for theorems 3.5 and 3.6 is provided in [3].

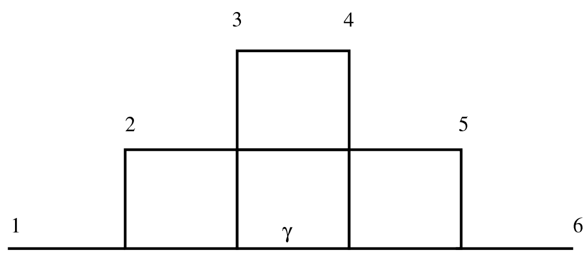
It is clear, then, that the response matrices of graphs containing negative conductors cannot have positive network equivalents:

Corollary 3.7. *Let (Γ, γ) be a critical, recoverable, circular planar network containing at least one negative conductor, and let Λ_γ be the corresponding response matrix. Then there is no positive circular planar network (Γ', γ') such that $\Lambda_{\gamma'} = \Lambda_\gamma$.*

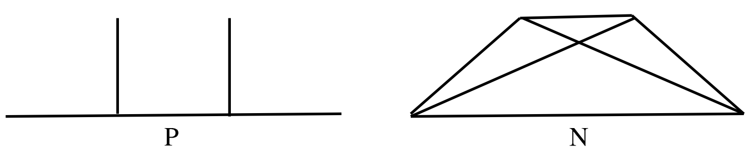
Proof. Suppose otherwise. Since Γ is a critical graph, by theorem 3.6, there exist positive conductances γ' on Γ such that Λ is the response matrix for (Γ, γ'). However, since Γ is recoverable, there is exactly one set of conductances on Γ that will yield the response matrix Λ. So $\gamma = \gamma'$. But this cannot be, since γ' is all positive, and γ is known to have at least one negative value.

3.2. Equivalence between mixed-sign and non-planar networks

However, it is possible to find mixed-sign conductor networks whose response matrix is equivalent to that of a non-planar graph. Consider, for example, the graph Σ_6:



Depending on the values of neighboring conductors, the conductor marked γ can be taken to have any of a range of negative conductances, $-\infty < \gamma < \gamma_0 < 0$, and the resulting response matrix Λ may still lie in K^6, the space of 6-dimensional Kirkhoff matrices of positive conductance. To see this, we may remove γ and its neighboring conductors, and replace them with a non-planar but electrically equivalent object:



For each of the objects N and P, we can construct a response matrix Λ_N and Λ_P respectively; since these objects are supposed to be electrically equivalent, we then have $\Lambda_N = \Lambda_P$. In this case, we have

$$\Lambda_P = \frac{1}{(a+b+c)(c+d+e)-c^2} \begin{pmatrix} \Sigma & -ab(c+d+e) & -acd & -ace \\ -ab(c+d+e) & \Sigma & -bcd & -bce \\ -acd & -bcd & \Sigma & -de(a+b+c) \\ -ace & -bce & -de(a+b+c) & \Sigma \end{pmatrix}$$

Since the graph N is fully connected, we can simply read the conductances from the response matrix. If all of the entries are to have the right sign when $c < 0$, then we must also have $a+b+c < 0$, $c+d+e < 0$, and $(a+b+c)(c+d+e)-c^2<0$, i.e., $c<-\frac{(a+b)(d+e)}{a+b+d+e}$. However, this last condition is redundant, since if $x,y>0$, then $\frac{xy}{x+y}<x+y$; thus, the appropriate condition is

$$c+\max\{a+b,d+e\}<0$$

This method is easily generalizable to arbitrary graphs.

Theorem 3.8. *Suppose that we are given a graph (Γ,γ), and a pair of nodes A and B in Γ such that A and B are connected by a single conductor with conductance $\gamma_{AB}<0$, but that all other conductances are positive; and denote by $N(x)$ the set of all nodes joined to x through a single conductor. Denote by Σ_A the sum of all conductances at A, and likewise define Σ_B. Then the response matrix $\Lambda_\gamma\in K$ provided that*

$$\sum_{p\in N(A)}\gamma_{pA}<0 \tag{3.8.1}$$

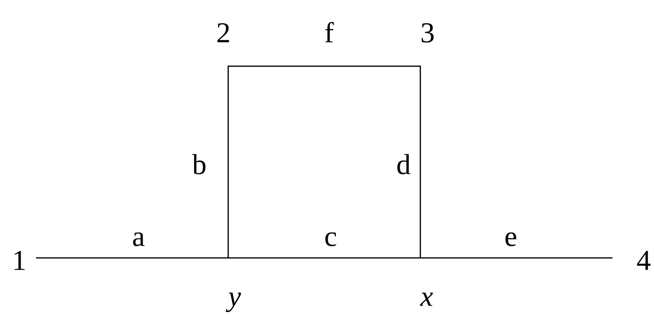
$$\sum_{q\in N(B)}\gamma_{qA}<0 \tag{3.8.2}$$

and

$$\gamma_{AB}<\frac{\Sigma_A\Sigma_B}{\Sigma_A+\Sigma_B} \tag{3.8.3}$$

Proof. Let Γ' be the set of nodes $N(A)\cup N(B)$, together with all edges joining those nodes to each other, and consider the response matrix $\Lambda_{\Gamma'}$. If node $p\in N(A)$, $q\in N(A)$, $p\neq q$, then $\lambda_{\Gamma':p,q}=-\frac{\gamma_{pA}\gamma_{pB}\Sigma_B}{\Sigma_A\Sigma_B-\gamma_{AB}^2}$; if $q\in N(A)$ as well, $\lambda_{\Gamma':p,q}=-\frac{\gamma_{pA}\gamma_{pB}\gamma_{AB}}{\Sigma_A\Sigma_B-\gamma_{AB}^2}$. If $\gamma_{AB}<0$, the off-diagonal entries will have mixed signs unless (3.8.1) and (3.8.2) hold; and the sign of the entries in the upper-right and lower-left blocks will be wrong unless (3.8.3) holds. However, if those conditions are satisfied, $\Lambda_{\Gamma'}$ will be a viable response matrix; since the rest of Γ has positive conductance, this means that Λ_Γ is also a viable response matrix, i.e., $\Lambda_\Gamma\in K$.

Lemma 3.9. *Suppose that $M\in K^4$. Then there exists a set of conductances γ on Σ_4 such that $M=\Lambda_\gamma$.*



Proof. Let Σ_4 be labelled as in the illustration. We can use a standard recovery technique to attempt to recover the conductances γ:

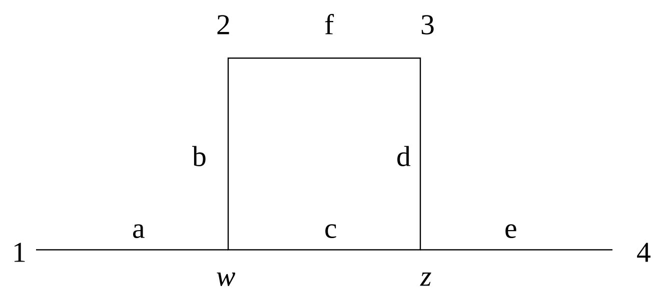
$$x = -m_{14}/m_{13} \tag{3.9.1}$$

$$\gamma_e = m_{43}x + m_{44} \tag{3.9.2}$$

$$\gamma_d = m_{43} + m_{44}/x \tag{3.9.3}$$

$$\gamma_f = m_{23} + m_{24}/x \tag{3.9.4}$$

We can see that in order to solve for γ_d, γ_e, and γ_f, we must have certain conditions on M; namely, $m_{13} \neq 0$, from (3.9.1), and $m_{14} \neq 0$, from (3.9.3) and (3.9.4). Similarly, recovering the left-hand ladder of conductors, we find the conditions $m_{42} \neq 0$ and $m_{41} \neq 0$. However, since we know already that $m_{ij} \neq 0$, all of these conditions are met, and we can solve uniquely for $\gamma_a, \gamma_b, \gamma_d, \gamma_e, \gamma_f$. Solving now for γ_c, we place a potential of $+1$ at the leftmost boundary node, and 0 at all other boundary nodes; then,



$$w = m_{11}/\gamma_a \tag{3.9.5}$$

$$z = m_{14}/\gamma_e \tag{3.9.6}$$

$$\gamma_c = (m_{11} - m_{12})/(w - z) \tag{3.9.7}$$

(3.9.5) implies that $m_{14}(m_{12}+m_{11}) \neq 0$; since we already know that $m_{14} \neq 0$, this simply means that $m_{12} + m_{11} \neq 0$. However, the only way that this could be false would be if $m_{13} = m_{14} = 0$, which cannot be true. Similarly for the second condition: $m_{14}(m_{43}+m_{44}) \neq 0$ is true, since $m_{14} \neq 0$ and since M is a Kirkhoff matrix. To find γ_c, we need to know that $x - y \neq 0$. If $x - y = 0$, no current could flow through the conductor c regardless of its conductance; similarly, no current could flow through conductor f, regardless of its conductance. Therefore, $m_{13} = -m_{14}$, which cannot be true.

Once we see that these conditions are satisfied, the fomulas provide γ.

Conjecture 3.10. *Suppose that M is a $n \times n$ Kirkhoff matrix, $n > 1$, such that $m_{ij} \neq 0$ for all i, j. Then there exists a set of conductances γ on Σ_n such that $M = \Lambda_\gamma$.*

Proof of this conjecture is trivial for $n < 4$, and as above for $n = 4$; although experimental evidence seems to indicate that the set of Kirkhoff matrices for which no Σ_n graphs can be found is small, no proof is offered for $n > 4$.

3.3. Future directions

If the last conjecture proves correct, it will provide an interesting relationship between planar, mixed-sign networks and non-planar, positive networks; it may be possible, if this is true, to discover the genus number of the embedding Riemann surface by reading the values gleaned from a recovery of Σ_n, or by other, more direct means. In order to prove the conjecture, it should be sufficient to prove it for a small case (say $n = 4$) and prove the rest by induction; however, time had not permitted such a proof to be formulated.

In providing a relationship between the recovered Σ_n conductances and the genus number of the embedding surface, it would also be necessary to provide a more general substitution proof than theorem 3.8; the experimental data, several thousand randomly generated 6×6 Kirkhoff matrices, gave rise to Σ_6 networks with as many as four negative conductors, in arbitrary locations on the graph; theorem 3.8 was simply not effective in these cases.

References

[1] Curtis, Edward B. and Morrow, James A., Determining the Resistors in a Network. SIAM J. Appl. Math, Vol. 50, No. 3, pp. 918-930, June 1990.

[2] Curtis, Edward B. and Morrow, James A., The Dirichlet to Neumann Map for a Resistor Network. SIAM J. Appl. Math, Vol. 51, No. 4, pp. 1011-1029, August 1991.

[3] Curtis, E. B., Ingerman, David, and Morrow, J. A. Circular Planar Graphs and Resistor Networks. [prepublication]