Recovery of networks from Λ and Φ maps

Aleksandr Murkes

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1 Introduction

We consider a network of conductors Ω, with set of nodes denoted by Ω_n, set of boundary nodes denoted by Ω_b, and set of interior nodes denoted by Ω_i, so that

$$\Omega_n = \Omega_b \cup \Omega_i.$$

For such a network, the Kirchoff matrix, the Λ map, and the Φ map are defined (for background and definitions see [1]).

We will divide a network into levels. The set of nodes for each level is denoted L_i, where i is a positive integer indicating the number of level. $\mathrm{N}(\mathrm{L}_i)$ will denote cardinality of L_i. Let the first level be the boundary nodes. Let a node belong to level L_{i+1} if it is connected to a node in level L_i, and if it has not yet been assigned a level.

We will number nodes of a network Ω so that nodes of the first level go first, nodes of the second level go second, and so on. Now we can write $K(\Omega)$ and $\Phi(\Omega)$ in the following block form:

$$K(\Omega) = \begin{array}{|c|c|c|} \hline K' & B^T & 0 \\ \hline B & C & D^T \\ \hline 0 & D & A \\ \hline \end{array} \quad \text{and} \quad \Phi(\Omega) = \begin{array}{|c|} \hline I \\ \hline P \\ \hline X \\ \hline \end{array}$$

where P contains information about potentials on the second level, and X contains information about potentials on the rest of the interior nodes. Using Kirchoff's and Ohm's Laws we obtain the following matrix equation:

$$\begin{array}{|c|c|c|}\hline K' & B^T & \mathit{0} \\ \hline B & C & D^T \\ \hline \mathit{0} & D & A \\ \hline \end{array} \quad \begin{array}{|c|}\hline I \\ \hline P \\ \hline X \\ \hline \end{array} = \begin{array}{|c|}\hline \Lambda(\Omega) \\ \hline \mathit{0} \\ \hline \mathit{0} \\ \hline \end{array} \tag{1}$$

where $\mathit{0}$ is the zero matrix.

In this paper we will look at networks with the following properties:

1. If level L_i is not the last level, it contains at least four nodes.
2. For each node in level L_i there is a node in level L_i to which it is not connected.
3. Each node in L_i is connected to exactly one node in L_{i+1}.
4. For every two nodes in L_i there has to be a path between them that doesn't go through outer levels.

We will call any network that has these properties a *well-leveled* network. We will show that any well-leveled network is recoverable from given $\Lambda(\Omega)$ and $\Phi(\Omega)$ maps.

2 Recovery of The First Level

¿From Equation (1) we have:

$$K'I + B^T P + \mathit{0}X = K' + B^T P = \Lambda.$$

Property 3 of well-leveled networks tells us that matrix B^T contains exactly one non-zero element in each row. Let $b_{i,g}$ be the non-zero element in the ith row of B^T. Property 2 says that in each row of

K' there is at least one zero. In other words, for each fixed i there is such j that

$$K'_{i,j} = 0,\ b_g P_{g,j} = \Lambda_{i,j}.$$

Thus, we are able to find B^T, and then

$$K' = \Lambda - B^T P.$$

3 Potentials on The Second Level

Let p be a node in L_{i+1}, and let $\mathrm{Q}(p)$ be the set of nodes in L_i that are connected to p. We will have $\mathrm{N}(\mathrm{L}_{i+1})$ such sets, every set containing at least one element. We will pick one element from each of these sets, and let these chosen elements form another set $\mathrm{W}_i \in \mathrm{L}_i$.

Claim: If we put zero potentials on every node $d \in \mathrm{L}_i \backslash \mathrm{W}_i$, then any desired combination of potentials on L_{i+1} uniquely determines the potentials on W_i.

Proof: When numbering nodes of L_i, we will first number the nodes in W_i. We now rewrite Equation (1):

$$\begin{array}{|c|c|c|c|}\hline K_1' & K_3'^T & B_1^T & \mathit{0} \\ \hline K_3' & K_2' & B_2^T & \mathit{0} \\ \hline B_1 & B_2 & C & D^T \\ \hline \mathit{0} & \mathit{0} & D & A \\ \hline \end{array}\ \begin{array}{|c|c|}\hline I & * \\ \hline \mathit{0} & * \\ \hline P_1 & * \\ \hline X_1 & * \\ \hline \end{array} = \begin{array}{|c|c|}\hline \Lambda_1 & * \\ \hline \Lambda_2 & * \\ \hline \mathit{0} & * \\ \hline \mathit{0} & * \\ \hline \end{array} \tag{2}$$

Here * indicates that this information is not useful and can be ignored. Also, $P = \begin{array}{|c|c|}\hline P_1 & * \\ \hline \end{array}$, where P_1 is a square matrix. We partition K' and B to make B_1 of the same size as P_1. Then B_1 represents the connections from W_1 to L_2. From this we obtain the following equations:

$$B_1 I + B_2 0 + CP_1 + D^T X_1 = B_1 + CP_1 + D^T X_1 = 0 \tag{3}$$

$$0I + 0 + DP_1 + AX_1 = DP_1 + AX_1 = 0 \tag{4}$$

Notice that the second terms in (3) and (4) are eliminated because of multiplication by zero. If $\mathrm{N}(\mathrm{L}_1) = \mathrm{N}(\mathrm{L}_2)$, then those terms don't even exist, because in that case $P_1 = P$. Either way, we get the same equations. Now we rewrite (4)

$$X_1 = -A^{-1} D P_1,$$

and substituting X_1 into (3), and rewriting (3), we have:

$$(C - D^T A^{-1} D) P_1 = -B_1.$$

Because every node in W is connected to exactly one node in L_2, matrix B_1 has exactly one element in each row and each column, and, therefore, it is non-singular. This makes P_1 a non-singular matrix. — **Q.E.D.**

Let Ω' be the subnetwork of Ω such that

$$\Omega' = \Omega \backslash \mathrm{L}_1, \text{ and } \Omega'_b = \mathrm{L}_2.$$

Clearly, Ω' is a well-leveled network. We would like to compute $\Phi(\Omega')$. Consider the following matrix product:

$$\Phi(\Omega) \cdot P_1^{-1} = \begin{array}{|c|} \hline I \\ \hline 0 \\ \hline P_1 \\ \hline X_1 \\ \hline \end{array} \; P_1^{-1} \; = \begin{array}{|c|} \hline P_1^{-1} \\ \hline 0 \\ \hline I \\ \hline X_1 P_1^{-1} \\ \hline \end{array}$$

From this we can see that

$$\Phi(\Omega') = \begin{array}{|c|}\hline I \\ \hline X_1 P_1^{-1} \\ \hline \end{array}$$

and

$$\begin{array}{|c|}\hline P_1^{-1} \\ \hline 0 \\ \hline \end{array}$$

is what what we need to put on L_1 in order to get I on L_2.

We can also find $\Lambda(\Omega')$. Let i be a node in L_2, and let j be another node in L_2. We will put potential of 1 on node i, and zeros on other nodes of L_2. Remember that $L_2 = \Omega'_b$. Then the boundary current at node j due to this combination of potentials is the current flowing through the conductors connecting j and L_1. Therefore, the matrix $B_1^T P_1^{-1}$ differs from $\Lambda(\Omega')$ only on the diagonal. Since each diagonal entry in a Λ matrix is the opposite of the sum of other entries in the same row, $\Lambda(\Omega')$ can be easily found.

Thus, we can use the same technique and recover the first level of Ω', and then consider its subnetwork, and so on, eventually recovering the entire original network.

4 An Example of Recovering A Well-Leveled Network

In the following example we will show how in a well-leveled network we can recover conductors between the nodes of the first and second levels, and between the nodes of the first level. Figure 1 shows part of a simple well-leveled network Ω.

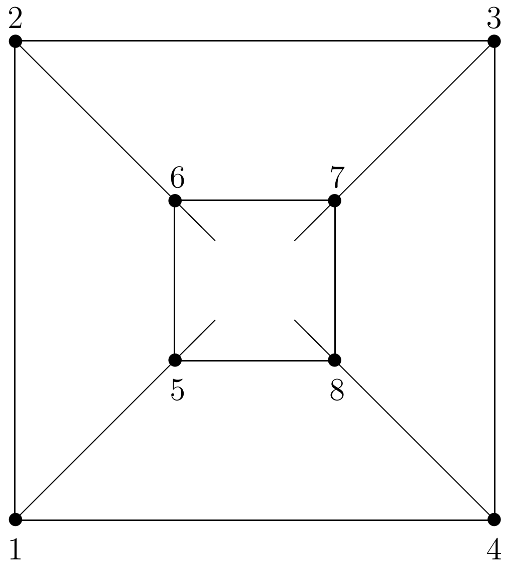

Figure 1

Nodes 1, 2, 3 and 4 are boundary nodes. Therefore, they form the first level L_1. Nodes in L_1 are connected to nodes 5, 6, 7 and 8, which in turn form L_2. We do not care about other levels of the network at this point. Let *conductor (i,j)* denote the conductor that connects nodes i and j.

We put potential of 1 on node 1, and zeros on other nodes of L_1. Then the current at node 3 is $\Lambda_{3,1}$, and it is flowing only through the conductor (3, 7). Since nodes 1 and 3 are not connected, $K_{3,1} = 0$, thus

$$\gamma_{3,7} = -K_{3,7} = -\frac{\Lambda_{3,1}}{\Phi_{7,1}}$$

Similarly, we can find $\gamma_{1,5}$, $\gamma_{2,6}$, and $\gamma_{4,8}$.

Now, we will put potential of 1 on node 1, and zeros on other nodes of L_1. The current in node 2 will be flowing through the conductors (2,1) and (2,6). We already know $K_{2,6}$, so

$$\gamma_{2,1} = -K_{2,1} = -(\Lambda_{2,1} - K_{2,6}\Phi_{6,1}).$$

Again, similarly we can find other boundary connections. Then we can find $\Lambda(\Omega')$ and $\Phi(\Omega')$ as described in the previous section, and recover the first level of Ω' in a similar fashion.

5 Relationships Between Elements of Λ and Φ matrices

A well-leveled network contains fewer conductors than there are elements in Λ and Φ maps. Therefore, there must exist certain relationships between these pieces of information.

Consider a network Ω. Put potentials of 1 on every boundary node. Then potentials on the interior nodes will be 1. This means that the sum of every row of $\Phi(\Omega)$ matrix is 1. Notice that this is true for any network.

Now, consider the well-leveled network Ω shown in Figure 2.

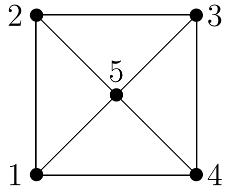

Figure 2

Nodes 1, 2, 3 and 4 are boundary. We will have the following matrix equation:

$$\begin{pmatrix} * & -\gamma_{1,2} & 0 & -\gamma_{1,4} & -\gamma_{1,5} \\ -\gamma_{2,1} & * & -\gamma_{2,3} & 0 & -\gamma_{2,5} \\ 0 & -\gamma_{3,2} & * & -\gamma_{3,4} & -\gamma_{3,5} \\ -\gamma_{4,1} & 0 & -\gamma_{4,3} & * & -\gamma_{4,5} \\ * & * & * & * & \Sigma \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ v_1 & v_2 & v_3 & v_4 \end{pmatrix} =$$

$$\begin{pmatrix} * & -\gamma_{1,2} - \gamma_{1,5}v_2 & -\gamma_{1,5}v_3 & -\gamma_{1,4} - \gamma_{1,5}v_4 \\ * & * & -\gamma_{2,3} - \gamma_{2,5}v_3 & -\gamma_{2,5}v_4 \\ * & * & * & -\gamma_{3,4} - \gamma_{3,5}v_4 \\ * & * & * & * \end{pmatrix},$$

where * indicates any entry that is dependent on others and can be ignored. Also, $\Sigma = (-\gamma_{1,5} - \gamma_{2,5} - \gamma_{3,5} - \gamma_{4,5})$. Since the potential inside is the weighted average of boundary potentials,

$$v_1 = \frac{\gamma_{1,5}}{\Sigma},$$
$$v_2 = \frac{\gamma_{2,5}}{\Sigma},$$
$$v_3 = \frac{\gamma_{3,5}}{\Sigma},$$
$$v_4 = \frac{\gamma_{4,5}}{\Sigma}.$$

Now,

$$\frac{v_1 v_3}{v_2 v_4} = \frac{\frac{\gamma_{1,5}\gamma_{3,5}}{\Sigma^2}}{\frac{\gamma_{2,5}\gamma_{4,5}}{\Sigma^2}} = \frac{\gamma_{1,5}\gamma_{3,5}}{\gamma_{2,5}\gamma_{4,5}},$$

and

$$\frac{\lambda_{1,3}}{\lambda_{2,4}} = \frac{\frac{-\gamma_{1,5}\gamma_{3,5}}{\Sigma}}{\frac{-\gamma_{2,5}\gamma_{4,5}}{\Sigma}} = \frac{\gamma_{1,5}\gamma_{3,5}}{\gamma_{2,5}\gamma_{4,5}}.$$

Thus,

$$\frac{v_1 v_3}{v_2 v_4} = \frac{\lambda_{1,3}}{\lambda_{2,4}}.$$

Out of 10 seemingly independent pieces of information we are left with only 8, since we found two relationships. This is equal to the number of conductors in the network.

It appears that when searching for similar dependencies in larger well-leveled networks, we should examine entries $\Lambda_{i,j}$ such that $K_{i,j} = 0$.

6 Acknowledgements

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7 References

[1] David Ingerman. "Theory of Equivalent Networks and Some of its Applications"