Recovery of networks from 
$\Lambda$ and $\Phi$ maps

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1 Introduction

We consider a network of conductors $\Omega$, with set of nodes denoted by $\Omega_n$, set of boundary nodes denoted by $\Omega_b$, and set of interior nodes denoted by $\Omega_i$, so that

$$\Omega_n = \Omega_b \cup \Omega_i.\)$$

For such a network, the Kirchoff matrix, the $\Lambda$ map, and the $\Phi$ map are defined (for background and definitions see [1]).

We will divide a network into levels. The set of nodes for each level is denoted $L_i$, where $i$ is a positive integer indicating the number of level. $N(L_i)$ will denote cardinality of $L_i$. Let the first level be the boundary nodes. Let a node belong to level $L_{i+1}$ if it is connected to a node in level $L_i$, and if it has not yet been assigned a level.

We will number nodes of a network $\Omega$ so that nodes of the first level go first, nodes of the second level go second, and so on. Now we can write $K(\Omega)$ and $\Phi(\Omega)$ in the following block form:

$$K(\Omega) = \begin{bmatrix} K' & B^T & 0 \\ B & C & D^T \\ 0 & D & A \end{bmatrix} \quad \text{and} \quad \Phi(\Omega) = \begin{bmatrix} I \\ P \\ X \end{bmatrix}$$
where $P$ contains information about potentials on the second level, and $X$ contains information about potentials on the rest of the interior nodes. Using Kirchoff’s and Ohm’s Laws we obtain the following matrix equation:

\[
\begin{bmatrix}
K' & B^T & 0 \\
B & C & D^T \\
0 & D & A
\end{bmatrix}
\begin{bmatrix}
I \\
P \\
X
\end{bmatrix}
= \begin{bmatrix}
\Lambda(\Omega) \\
0 \\
0
\end{bmatrix}
\]  \hspace{1cm} (1)

where $0$ is the zero matrix.

In this paper we will look at networks with the following properties:

1. If level $L_i$ is not the last level, it contains at least four nodes.
2. For each node in level $L_i$ there is a node in level $L_i$ to which it is not connected.
3. Each node in $L_i$ is connected to exactly one node in $L_{i+1}$.
4. For every two nodes in $L_i$ there has to be a path between them that doesn’t go through outer levels.

We will call any network that has these properties a well-leveled network. We will show that any well-leveled network is recoverable from given $\Lambda(\Omega)$ and $\Phi(\Omega)$ maps.

## 2 Recovery of The First Level

From Equation (1) we have:

\[
K'I + B^TP + 0X = K' + B^TP = \Lambda.
\]

Property 3 of well-leveled networks tells us that matrix $B^T$ contains exactly one non-zero element in each row. Let $b_{i,g}$ be the non-zero element in the $i$th row of $B^T$. Property 2 says that in each row of
there is at least one zero. In other words, for each fixed $i$ there is such $j$ that

$$K'_{i,j} = 0, \quad b_g P_{g,j} = \Lambda_{i,j}.$$  

Thus, we are able to find $B^T$, and then

$$K' = \Lambda - B^T P.$$

3 Potentials on The Second Level

Let $p$ be a node in $L_{i+1}$, and let $Q(p)$ be the set of nodes in $L_i$ that are connected to $p$. We will have $N(L_{i+1})$ such sets, every set containing at least one element. We will pick one element from each of these sets, and let these chosen elements form another set $W_i \in L_i$.

**Claim:** If we put zero potentials on every node $d \in L_i \setminus W_i$, then any desired combination of potentials on $L_{i+1}$ uniquely determines the potentials on $W_i$.

**Proof:** When numbering nodes of $L_i$, we will first number the nodes in $W_i$. We now rewrite Equation (1):

$$K_1' K_3'^T B_1^T 0 I * = \Lambda_1 *$$

$$K_3' K_2' B_2^T 0 0 * = \Lambda_2 *$$

$$B_1 \ B_2 \ C \ D^T P_1 * 0 *$$

$$0 \ 0 \ D \ A X_1 * 0 *$$

Here * indicates that this information is not useful and can be ignored. Also, $P = \begin{bmatrix} P_1 & * \\ * & * \end{bmatrix}$, where $P_1$ is a square matrix. We partition $K'$ and $B$ to make $B_1$ of the same size as $P_1$. Then $B_1$ represents the connections from $W_1$ to $L_2$. From this we obtain the following equations:
\[ B_1 I + B_2 \theta + CP_1 + D^T X_1 = B_1 + CP_1 + D^T X_1 = 0 \quad (3) \]
\[ 0 I + 0 + DP_1 + AX_1 = DP_1 + AX_1 = 0 \quad (4) \]

Notice that the second terms in (3) and (4) are eliminated because of multiplication by zero. If \( N(L_1) = N(L_2) \), then those terms don’t even exist, because in that case \( P_1 = P \). Either way, we get the same equations. Now we rewrite (4)

\[ X_1 = -A^{-1} DP_1, \]

and substituting \( X_1 \) into (3), and rewriting (3), we have:

\[ (C - D^T A^{-1} D)P_1 = -B_1. \]

Because every node in \( W \) is connected to exactly one node in \( L_2 \), matrix \( B_1 \) has exactly one element in each row and each column, and, therefore, it is non-singular. This makes \( P_1 \) a non-singular matrix. \( \text{— Q.E.D.} \)

Let \( \Omega' \) be the subnetwork of \( \Omega \) such that

\[ \Omega' = \Omega \setminus L_1, \text{ and } \Omega'_b = L_2. \]

Clearly, \( \Omega' \) is a well-leveled network. We would like to compute \( \Phi(\Omega') \). Consider the following matrix product:

\[
\Phi(\Omega) \cdot P_1^{-1} = \begin{bmatrix} I & P_1^{-1} \\ 0 & 0 \\ P_1 & I \\ X_1 & X_1 P_1^{-1} \end{bmatrix}
\]
From this we can see that

$$\Phi(\Omega') = \begin{bmatrix} I \\ X_1 P_1^{-1} \end{bmatrix}$$

and

$$\begin{bmatrix} P_1^{-1} \\ 0 \end{bmatrix}$$

is what we need to put on $L_1$ in order to get $I$ on $L_2$.

We can also find $\Lambda(\Omega')$. Let $i$ be a node in $L_2$, and let $j$ be another node in $L_2$. We will put potential of 1 on node $i$, and zeros on other nodes of $L_2$. Remember that $L_2 = \Omega_b'$. Then the boundary current at node $j$ due to this combination of potentials is the current flowing through the conductors connecting $j$ and $L_1$. Therefore, the matrix $B_1^T P_1^{-1}$ differs from $\Lambda(\Omega')$ only on the diagonal. Since each diagonal entry in a $\Lambda$ matrix is the opposite of the sum of other entries in the same row, $\Lambda(\Omega')$ can be easily found.

Thus, we can use the same technique and recover the first level of $\Omega'$, and then consider its subnetwork, and so on, eventually recovering the entire original network.

4 An Example of Recovering A Well-Leveled Network

In the following example we will show how in a well-leveled network we can recover conductors between the nodes of the first and second levels, and between the nodes of the first level. Figure 1 shows part of a simple well-leveled network $\Omega$. 
Nodes 1, 2, 3 and 4 are boundary nodes. Therefore, they form the first level $L_1$. Nodes in $L_1$ are connected to nodes 5, 6, 7 and 8, which in turn form $L_2$. We do not care about other levels of the network at this point. Let conductor $(i,j)$ denote the conductor that connects nodes $i$ and $j$.

We put potential of 1 on node 1, and zeros on other nodes of $L_1$. Then the current at node 3 is $\Lambda_{3,1}$, and it is flowing only through the conductor $(3, 7)$. Since nodes 1 and 3 are not connected, $K_{3,1} = 0$, thus

$$\gamma_{3,7} = -K_{3,7} = -\frac{\Lambda_{3,1}}{\Phi_{7,1}}$$

Similarly, we can find $\gamma_{1,5}$, $\gamma_{2,6}$, and $\gamma_{4,8}$.

Now, we will put potential of 1 on node 1, and zeros on other nodes of $L_1$. The current in node 2 will be flowing through the conductors $(2,1)$ and $(2,6)$. We already know $K_{2,6}$, so

$$\gamma_{2,1} = -K_{2,1} = -(\Lambda_{2,1} - K_{2,6}\Phi_{6,1}).$$

Again, similarly we can find other boundary connections. Then we can find $\Lambda(\Omega')$ and $\Phi(\Omega')$ as described in the previous section, and recover the first level of $\Omega'$ in a similar fashion.
5 Relationships Between Elements of $\Lambda$ and $\Phi$ matrices

A well-leveled network contains fewer conductors than there are elements in $\Lambda$ and $\Phi$ maps. Therefore, there must exist certain relationships between these pieces of information.

Consider a network $\Omega$. Put potentials of 1 on every boundary node. Then potentials on the interior nodes will be 1. This means that the sum of every row of $\Phi(\Omega)$ matrix is 1. Notice that this is true for any network.

Now, consider the well-leveled network $\Omega$ shown in Figure 2.

![Figure 2](image)

Nodes 1, 2, 3 and 4 are boundary. We will have the following matrix equation:

\[
\begin{pmatrix}
* & -\gamma_{1,2} & 0 & -\gamma_{1,4} & -\gamma_{1,5} \\
-\gamma_{2,1} & * & -\gamma_{2,3} & 0 & -\gamma_{2,5} \\
0 & -\gamma_{3,2} & * & -\gamma_{3,4} & -\gamma_{3,5} \\
-\gamma_{4,1} & 0 & -\gamma_{4,3} & * & -\gamma_{4,5} \\
* & * & * & * & \Sigma
\end{pmatrix}
\begin{pmatrix}
1 \\
0 \\
1 \\
0 \\
\Sigma
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4
\end{pmatrix}
= 
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4
\end{pmatrix},
\]

where * indicates any entry that is dependent on others and can be ignored. Also, $\Sigma = (\gamma_{1,5} - \gamma_{2,5} - \gamma_{3,5} - \gamma_{4,5})$. Since the potential inside is the weighted average of boundary potentials,
\[
v_1 = \frac{\gamma_{1,5}}{\Sigma},
\]
\[
v_2 = \frac{\gamma_{2,5}}{\Sigma},
\]
\[
v_3 = \frac{\gamma_{3,5}}{\Sigma},
\]
\[
v_4 = \frac{\gamma_{4,5}}{\Sigma}.
\]

Now,
\[
\frac{v_1 v_3}{v_2 v_4} = \frac{\frac{\gamma_{1,5}\gamma_{3,5}}{\Sigma^2}}{\frac{\gamma_{2,5}\gamma_{4,5}}{\Sigma^2}} = \frac{\gamma_{1,5}\gamma_{3,5}}{\gamma_{2,5}\gamma_{4,5}},
\]
and
\[
\frac{\lambda_{1,3}}{\lambda_{2,4}} = \frac{\frac{-\gamma_{1,5}\gamma_{3,5}}{\Sigma}}{\frac{-\gamma_{2,5}\gamma_{4,5}}{\Sigma}} = \frac{\gamma_{1,5}\gamma_{3,5}}{\gamma_{2,5}\gamma_{4,5}}.
\]

Thus,
\[
\frac{v_1 v_3}{v_2 v_4} = \frac{\lambda_{1,3}}{\lambda_{2,4}}.
\]

Out of 10 seemingly independent pieces of information we are left with only 8, since we found two relationships. This is equal to the number of conductors in the network.

It appears that when searching for similar dependencies in larger well-leveled networks, we should examine entries \(\Lambda_{i,j}\) such that \(K_{i,j} = 0\).
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7 References

[1] David Ingerman. ”Theory of Equivalent Networks and Some of its Applications”