A Generalization into Three Dimensions of the Resistor Isolating Algorithm

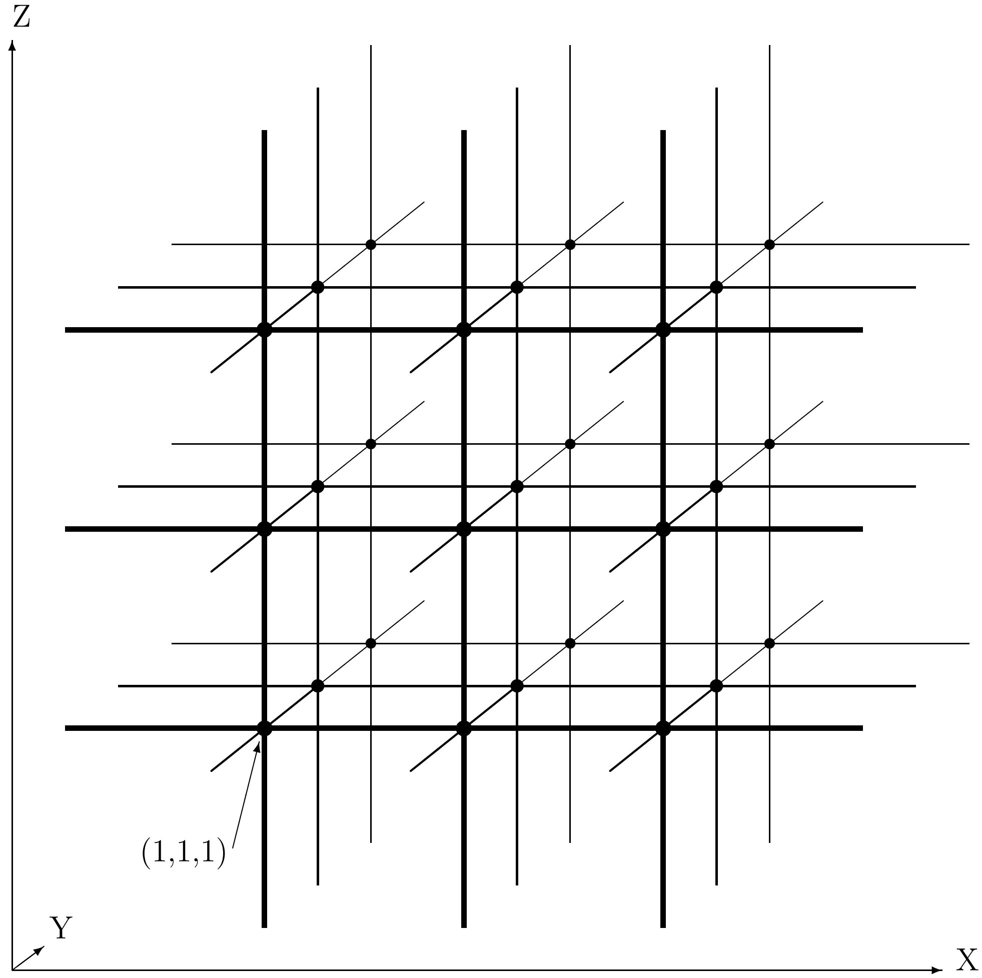
Matthew Munro

13 Aug. 1993

INTRODUCTION: Consider networks of resistors in a three dimensional lattice as shown in figure 1. Call the points at the ends of each resistor nodes. There are no current sources or sinks inside the lattice so Kirkhoff's law holds over the interior. Nodes where Kirkhoff's law holds are called interior nodes. Boundary nodes are those where Kirkhoff's law does not hold. At boundary nodes current can be measured and known voltages applied, referred to as boundary conditions. Two nodes are neighbors when they are connected directly by a resistor. In the three dimensional networks considered here all interior nodes have exactly six neighbors. Each boundary node has exactly one neighbor. The following conventions will be used to describe networks and nodes. Each network is made up of a latticed box of interior nodes connected by resistors with spikes leading from the nodes on the surface of the box to boundary nodes. The size of the box of interior nodes will be considered to be the size of the network so that figure 1 illustrates a 3x3x3 network. One arbitrary corner of the box will be considered to lie at the point (1,1,1) in Cartesian space. Thus, in a 3x3x3 network, boundary nodes lie in the planes x=1, x=4, y=1, y=4, z=1,and z=4, while the interior nodes lie at the intersections of the planes x=1, 2, or 3, y=1, 2, or 3, and z=1, 2 or 3. Internal resistors lie inside the box of interior nodes. They are not on the surface and are not spikes. The problem addressed is to recover the values of the internal resistors from boundary conditions without first computing the outer resistors.

Figure 1

3x3x3 network

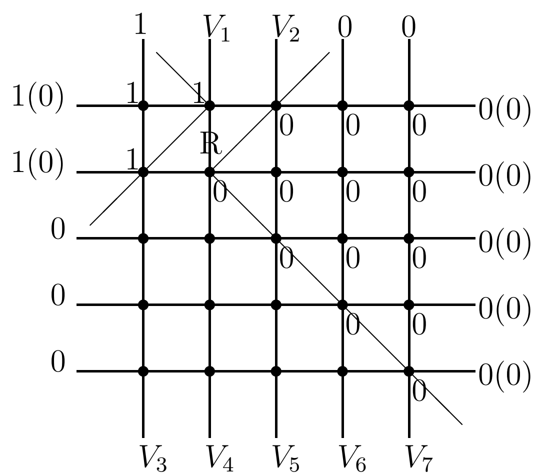


PREVIOUS RESULTS: Kirkhoff's law states that the net current through a closed region containing no current sources or sinks must be zero. This implies that, in the resistor networks considered, the net current through each interior node must be zero. Since current is a function of potential difference and resistance, this in turn implies that at an interior node the voltage is determined by the voltages of its neighbors weighted against the resistances between them. Also implied is that if a node and all but one of its neighbors have some determined voltage then that last neighbor must also have a determined voltage. [1]

The Matrix Λ for a given network is an NxN matrix where N is the number of boundary nodes, and $\lambda_{i,j}$ is the current out of node i when a voltage of 1 is applied at node j and voltage is 0 at all other nodes. [1]

It is possible in a two dimensional lattice network (figure 2) to create boundary conditions which impose certain conditions on the interior of the network which allow an interior resistor to be calculated directly without knowledge of the values of the other resistors. Essentially two wedges of zero current, one with voltage 1 and the other with voltage 0, have apexes at each end of an interior resistor. This implies that all current flowing through the resistor is the only current which flows between the two regions divided by the wedges. The current through the resistor is the sum of the currents into the boundary nodes in one or the other regions. Since the potential difference is one over the resistor, the resistance equals the inverse of the conductivity which equals the current. This has been named the resistor isolating algorithm. [2]

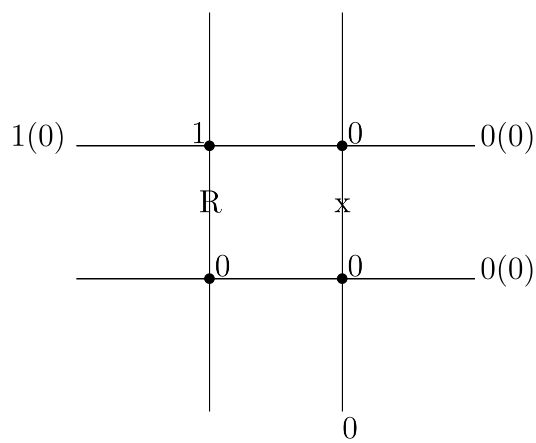
Figure 2



THE RESISTOR ISOLATING ALGORITHM: Consider a 2x2 two dimensional lattice network as shown in figure 3. In order to compute the resistor labeled R between nodes (1,1) and (1,2), place boundary conditions in the following way. A Voltage of 1 and zero current at node (0,2) forces a voltage of 1 at one end of the resistor. A Voltage of 0 and current of 0 at nodes (3,1) and (3,2) force a voltage of 0 on their interior neighbors, and a current of 0 through the resistor parallel to R. Now a voltage of 0 at node (2,0) forces a voltage of 0 on the other end of R. Now the wedges of 0 current are in place, and all current from the top of the network to the bottom must flow through R.

Figure 3

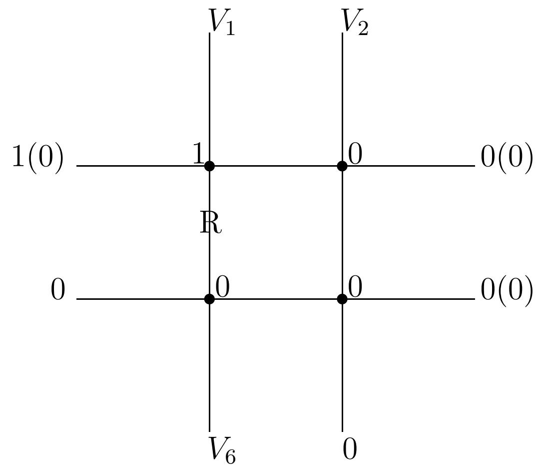
2x2 network



At nodes (1,2) and (2,2) voltages are now given, as are the voltages at all but one of the neighbors of each of those nodes. This uniquely determines the voltages at nodes (1,3) and (2,3). If node (0,1) is assigned a voltage of 0, then voltage at node (1,0) is uniquely determined as well. (Figure 4)

Figure 4

2x2 network



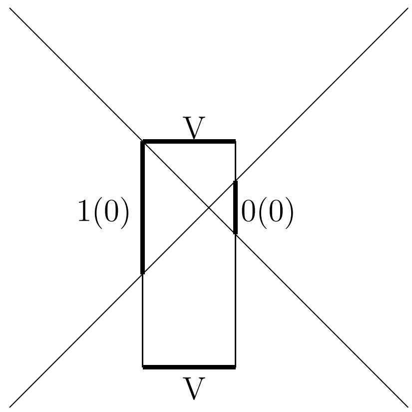
Thus it is proven that there is a unique set of values for V_1, V_2, and V_6 which along with the other given boundary voltages produce currents of 0 at nodes (0,2), (3,1), and (3,2). To find these values use the Λ matrix where the boundary nodes are numbered from 1 to 8, clockwise, beginning with node (1,3). So, for example, $\lambda_{1,2}$ is the current out of node (1,3) when voltage at node (2,3) is 1 and voltage at all other boundary nodes is 0. The current out of node (3,2), which must be 0, is given by $\lambda_{3,1}V_1 + \lambda_{3,2}V_2 + \lambda_{3,6}V_6 + \lambda_{3,8}$. Similar equations for the currents at nodes (3,1) and (0,2) yield a nonhomogeneous system of 3 equations with 3 unknowns.

$$\begin{bmatrix} \lambda_{3,1} & \lambda_{3,2} & \lambda_{3,6} \\ \lambda_{4,1} & \lambda_{4,2} & \lambda_{4,6} \\ \lambda_{8,1} & \lambda_{8,2} & \lambda_{8,6} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_6 \end{bmatrix} = \begin{bmatrix} -\lambda_{3,8} \\ -\lambda_{4,8} \\ -\lambda_{8,8} \end{bmatrix}$$

Since all boundary voltages are now known it's straight forward to compute the currents into each boundary node on one side of R, and then the resistance of R.

This algorithm will yield a square system of equations for any resistor between interior nodes in rectangular, 2 dimensional, lattice network. This geometric fact is illustrated in figure 5. No matter in what direction(s) the rectangle is expanded the total length of the line segments on the top and bottom sides, bounded by the upward and downward facing V's will always be equal to the total length of the segments on the right and left sides, bounded by the right and left facing V's.

Figure 5



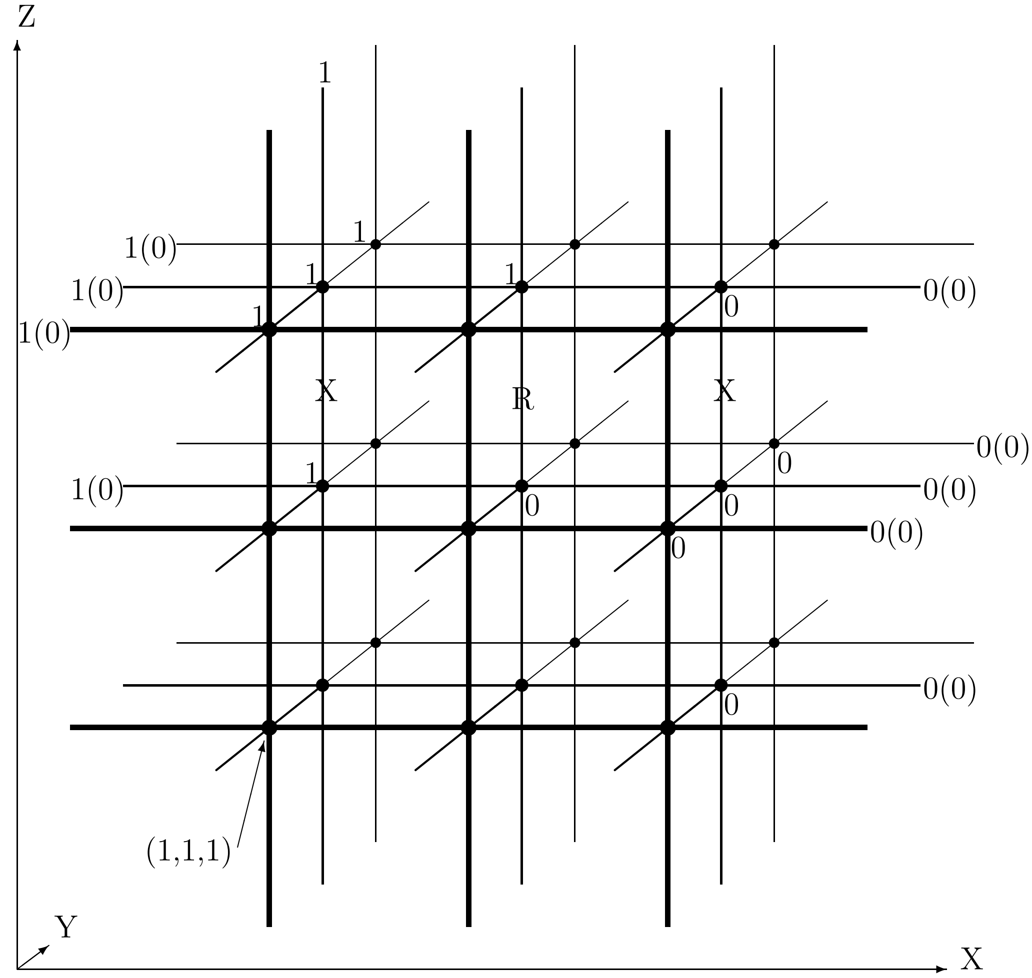
Returning to the network in figure 4, if the network is expanded by one layer in the positive x or y directions the algorithm will still yield a 3x3 system. Expansion in the negative x or y directions will yield a 4x4 system. Given any rectangular lattice and any internal resistor, figure 4 can be superimposed on that network with R over the desired resistor. Proceed by induction to expand the wedges until the necessary boundary conditions are known in order to isolate the resistor. Using the Λ matrix of the larger network a square system of equations can always be found and solved to yield the unique boundary voltages.

GENERALIZATION INTO THREE DIMENSIONS: As in the two dimensional case specific boundary conditions will determine the nature of the current flow in the interior of the network. Overlapping pyramids of 0 current will isolate an interior resistor such that the potential difference over that resistor is 1 and all current flow through a layer of the network will be only through the isolated resistor.

In the three dimensional case the algorithm will apply to any resistor such that not both nodes have boundary neighbors. The smallest network that has such a resistor is the 3x3x3 network pictured in figure 6. In order to calculate the resistance of R, begin by creating pyramids of 0 current with apexes at the ends of R as shown. Voltages of 0 and 0 currents at nodes (4,2,1), (4,1,2), (4,2,2), (4,3,2), and (4,2,3) force the 0(0) pyramid. Voltages of 1 and 0 currents at nodes (0,2,2), (0,1,3), (0,2,3), and (0,3,3) along with a voltage of 1 at node (1,2,4) force the 1(0) pyramid. Unlike the two dimensional case however, these boundary conditions are not sufficient to force all current through R. Only two of the resistors parallel to and through the same layer as R (marked with X's in figure 6) have no current flow. Boundary conditions are needed such that each resistor labeled with X's in figure 7 has no current flow.

Figure 6

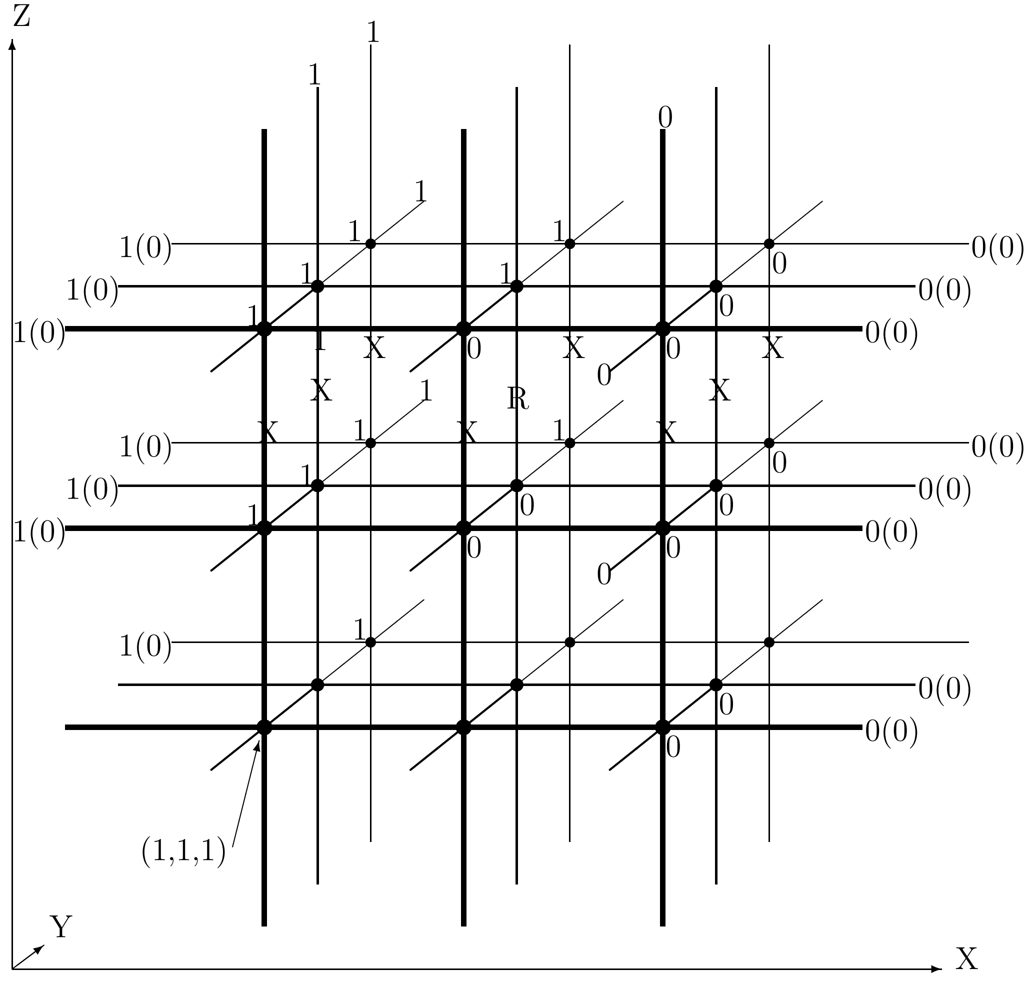
3x3x3 network



In order to finish isolating R, add boundary conditions as shown in figure 7; 0 voltages at nodes (3,1,4), (3,0,2), and (3,0,3), voltages of 1 at nodes (1,3,4), (1,4,3), and (1,4,2), voltages of 1 and currents of 0 at nodes (0,3,1), (0,1,2), and (0,3,2), and voltages of 0 with 0 current at nodes (4,1,1), (4,1,3), and (4,3,3). Now all current flowing from the upper part of the network to the lower must flow through R.

Figure 7

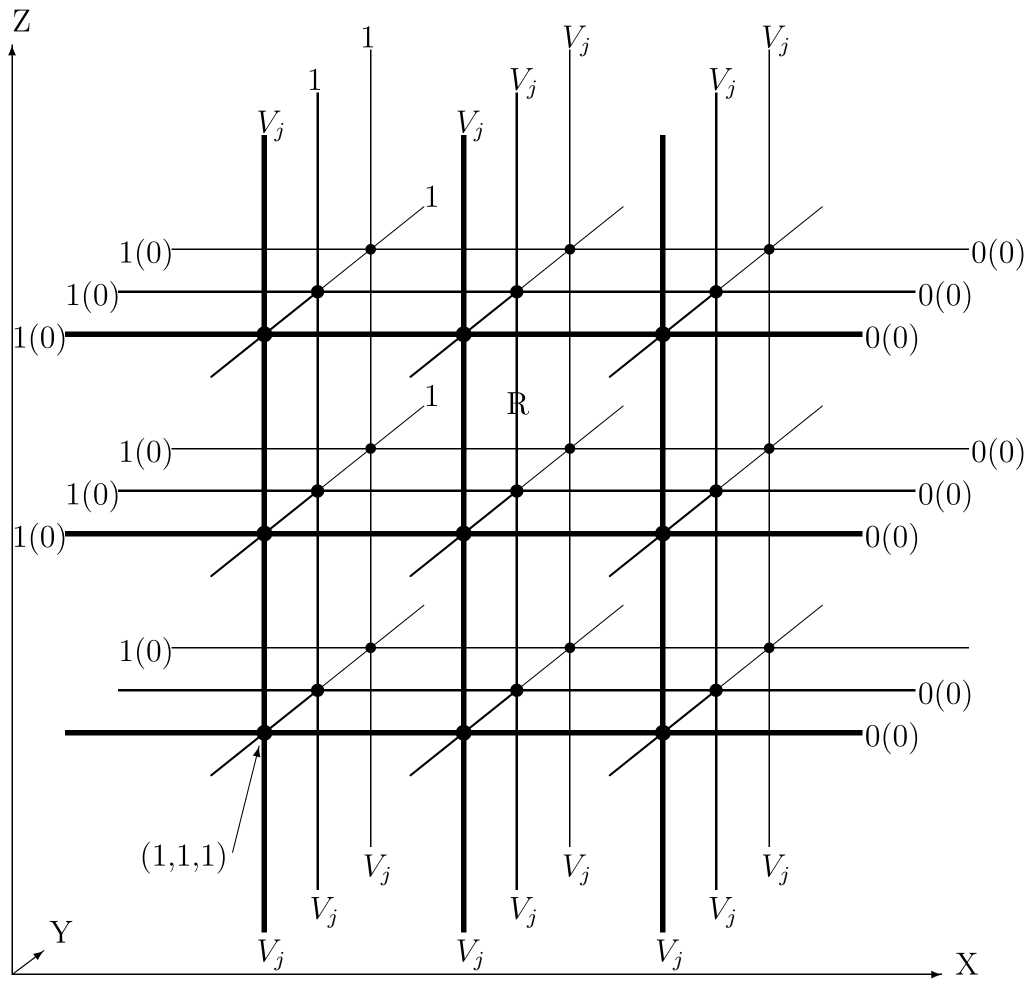
3x3x3 network



To complete the algorithm all remaining nodes on the x=0, x=4, y=0, and y=4 sides can arbitrarily be assigned a voltage of 0. This will determine some voltage at each of the remaining nodes on the z=0 and z=4 sides. The boundary conditions are now as in figure 8 with the unlabeled nodes all having 0 voltage.

Figure 8

3x3x3 network



The boundary nodes are numbered 1 to 54 for reference to the 54x54 Λ matrix. Now use Λ to develop the system of equations for the V_j. Let $J =$ The set of nodes at which voltages are to be determined. Let $I =$ The set of nodes at which current is 0. Let $M =$ The set of nodes at which voltage is 1. Then for all $i \in I$,

$\sum_{j \in J} \lambda_{i,j} V_j = -\sum_{m \in M} \lambda_{i,m}$.

Now that all the boundary voltages are known, Λ can be used to compute the net current into the boundary nodes above R with which the resistance is easily computed.

If the network is expanded by one layer in the negative z direction the algorithm will still yield a 15x15 system. An expansion of one in the positive x direction will lead to a 16x16 system. The negative x direction expansion produces a 17x17 system, and expansion in the positive y, positive z, or negative y directions yields an 18x18 system. Each additional expansion always increases the number of boundary nodes in the (0) pyramids and the V pyramids by the same amounts. Proceeding by induction shows that the algorithm will solve for any interior conductor in any three dimensional, lattice network.

References

[1] . B. Curtis and J. A. Morrow. *The Dirichlet to Neumann Map for a Resistor Network.* SIAM J. Appl. Math. Vol. 51, No. 4, pp 1011-1029, Aug. 1991.

[2] . Landrum. *A Comparison of Three Algorithms for the Inverse Conductivity Problem.* REU Report. 1990