Locating a Diode in a Rectangular Network Using Internal Current Sources

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I. Introduction

In this paper it is shown that the location and direction of a diode in the interior of a rectangular network can be determined by measurements of boundary output currents generated by internal current sources. The boundary measurements will also be used to determine the conductors (and conductances) in the network.

II. Preliminaries for all rectangular networks

We consider a rectangular network of conductors in R^2, as in the following figure:

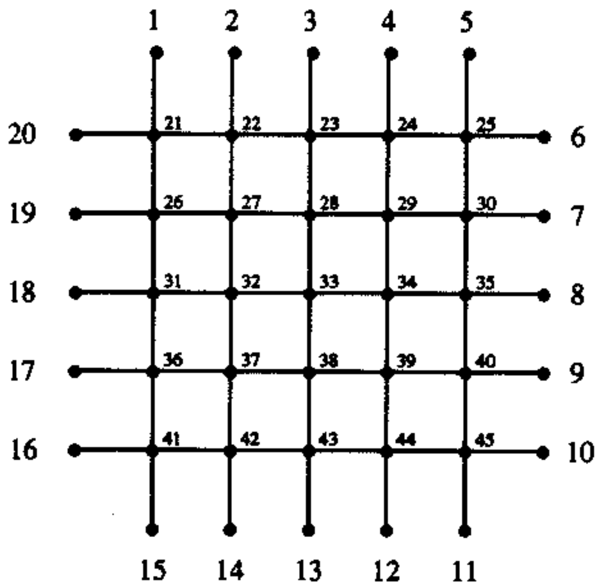


Figure 2.1

Such a network Ω with m-horizontal lines and n-vertical lines will be called a rectangular network of type $T(m,n)$. Figure 2.1 shows a rectangular network of type $T(5,5)$. For this paper, only square networks with n vertical & horizontal lines will be examined. However, the arguments presented will be applicable to any rectangular networks as well.

1. Network Definitions

2.1.1 The *nodes* of Ω, called Ω_0, are the end points of the line segments in the network.
2.1.2 The *boundary* of Ω, called $\partial\Omega_0$, consists of all nodes numbered from 1 to 4n. Each boundary node is connected to exactly one line segment.

Boundary nodes are numbered as follows: top row, left to right
right side, top to bottom
bottom row, right to left
left side, bottom to top.

2.1.3 The *interior* of Ω, called intΩ_0, consists of all nodes which are not boundary nodes, numbered from 4n+1 to n^2+4n. Each interior node is connected to more than one line segment. Let m equal the number of interior nodes, where $m = n^2$.
Interior nodes are numbered left to right, from the top row to the bottom row.
2.1.4 The *edges* of Ω, called Ω_1, consists of all conductors which connect two adjacent nodes. There exist 2n(n+1) edges in a square network with 4n boundary edges.. We assume each edge (pq) has a conductance value $\gamma_{pq}>0$.
2.1.5 The *neighbors* for each node $p \in \Omega_0$ are defined as the adjacent nodes which are connected to it by a conductor. (For each boundary node, there is exactly 1 neighbor, which is an interior node. For each interior node, there are exactly 4 neighbors.) Let N(p) be the set of all neighbors of node p.

Figure 2.1 shows a rectangular network with the following:

Ω_0, 45 - total nodes
$\partial\Omega_0$, 20 - boundary nodes (numbered 1 to 20)
intΩ_0, 25 - interior nodes (numbered 21 to 45)
Ω_1, 60 - edges (conductors)

2.1.6 The term *1-deep* will be used to describe the first "ring" of interior nodes in a network. This ring will consist of all interior nodes connected to boundary nodes. (Figure 2.2) The *2-deep* nodes will be the collection of nodes in the second ring of the network. (Figure 2.3) These nodes are connected to the 1-deep nodes. This notation will continue for *n-deep* nodes.

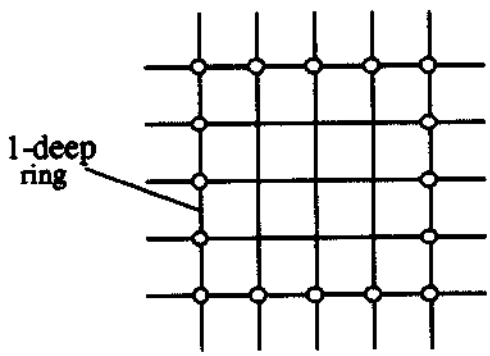


Figure 2.2

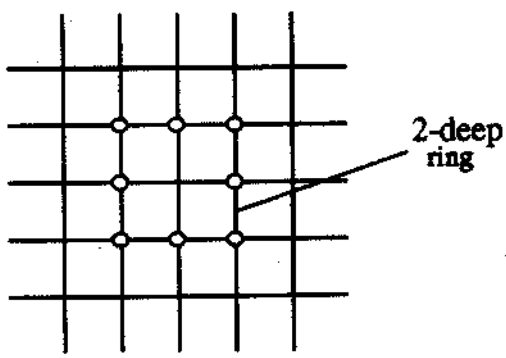


Figure 2.3

2.1.7 Let u be a real valued function on the network, called potential.
Ohm's Law states that the current from node p to an adjacent node q will equal the conductance from node p to node q multiplied by the potential difference from node p to node q, or simply $\mathbf{I_{pq}} = \gamma_{pq}(\mathbf{u_p(p)}-\mathbf{u_p(q)})$.

2.1.8 *Kirchhoff's Law* states that the total current flow into an interior node must equal the total current flow out of that node.

2.1.9 The *Maximum Principle* states that if a current is input at an interior node, and boundary nodes have potential of 0, then the potential at the source node must be larger than at any other node.

B. Network Properties

2.2.1 From the network, we construct a Kirchhoff matrix which has the form

$$K = \begin{bmatrix} \sigma_1 & -\gamma_{1,2} & \cdots & -\gamma_{1,i} & \cdots & -\gamma_{1,n} \\ -\gamma_{2,1} & \sigma_2 & \cdots & -\gamma_{2,i} & \cdots & -\gamma_{2,n} \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ -\gamma_{i,1} & -\gamma_{i,2} & \cdots & \sigma_i & \cdots & -\gamma_{i,n} \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ -\gamma_{n,1} & -\gamma_{n,2} & \cdots & -\gamma_{n,i} & \cdots & \sigma_n \end{bmatrix}.$$

Each σ_i is the sum of all γ_{ij} for nodes j connected to node i. For all $j \in N(i)$, the i,j entry of K will be the negative conductance along the edge (ij), i.e. $-\gamma_{ij}$. All other non-diagonal entries will be 0. This matrix has a block structure,

$$K = \left[\begin{array}{c|c} K' & C^T \\ \hline C & A \end{array}\right]$$

where
K' consists of boundary node to boundary node connections
C and C^T consist of boundary node to interior node connections, and
A consists of interior node to interior node connections.

2.2.2 The symmetry of A^{-1} is required in later sections of this paper, so the proof that A^{-1} is symmetric is as follows:

A is symmetric since $\gamma_{ij} = \gamma_{ji}$ for every i,j in a network.

$$\text{So,} \qquad A = A^T$$
$$\Rightarrow \quad A^{-1} = (A^T)^{-1}$$
$$\Rightarrow \quad A^{-1} = (A^{-1})^T$$

Therefore, A^{-1} is symmetric.

- For a complete understanding of the inverse diode problem, we must first investigate the characteristics of the forward problem without a diode. To do so, we have simulated experiments which produce the required information needed for the inverse problem. This process of simulating physical experiments is detailed below.

III. Forward problem without a diode

In a forward problem without a diode we are given γ_{pq}, where γ_{pq} is the value of the conductor between nodes p and q for all p,q $\in \Omega$. We then assume that the potential at all boundary nodes is 0. For each interior node i we will conduct a separate experiment to determine the potentials at every interior node. These experiments will be recorded in a vector of potentials, called u_i. We can then determine the current flows out of each boundary node.

- The potential at node q due to an input current of +1 at node p will be represented as $u_p(q)$. The potential at node q due to an input current of -1 at node p will be represented as $u_{-p}(q)$.

To solve for u_i, the vector of the potentials due to an input current of 1 at an interior node i, we will use the following system of equations: $\mathbf{A u_i = e_i}$, where

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ i \\ \vdots \\ 0 \end{bmatrix}.$$

The matrix obtained by using the e_i's as column vectors is

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} = \text{identity matrix}$$

The vectors u_i will be

$$u_1 = \begin{bmatrix} u_1(1) \\ u_1(2) \\ \vdots \\ u_1(4n) \end{bmatrix}, \quad u_2 = \begin{bmatrix} u_2(1) \\ u_2(2) \\ \vdots \\ u_2(4n) \end{bmatrix}, \quad \ldots\ldots, \quad u_m = \begin{bmatrix} u_m(1) \\ u_m(2) \\ \vdots \\ u_m(4n) \end{bmatrix},$$

where $u_i(j)$ is the potential at node j due to a source current at i. If we construct a matrix U by using the u_i's as column vectors we get $U = [\, u_1 ,\ u_2 ,\ \ldots\ldots ,\ u_m \,]$. Now U satisfies the equation $AU = I$. Therefore, $U = A^{-1}$.

• The following definitions are used to describe the notation for current flow across edges in the network.

Definition 3.1 Let node r be a boundary node. Input a source current of 1 at node s.
Then $f_s(r)$ is the current flow out of node r from its interior neighbor.

Definition 3.2 Let node r be an interior node, and node q be one of its interior neighbors.
Input a source current of 1 at node s.
Then $\phi_s(rq)$ is the current flow from node r to node q.

• For the next two definitions, the following holds: When two input sources are used, the first subscript always represents the node with an input current of 1. The second subscript always represents the node with an input current of α.

Definition 3.3 Let node r be a boundary node. Input a source current of 1 at node s, and a source current of α at node t.
Then $g_{s,t}(r)$ is the current flow out of node r from its interior neighbor.

Definition 3.4 Let node r be an interior node, and node q be one of its interior neighbors.
Input a source current of 1 at node s, and a source current of α at node t.
Then $\psi_{s,t}(rq)$ is the current flow from node r to node q.

Recall definition 3.1. Let f_i be the vector of currents flowing out of the boundary nodes, due to u_i. Next let F be the (4n x m) matrix of outflow vectors whose columns are $f_1, f_2, \ldots, f_m$.

We can calculate the matrix F by

$$-F = C^T A^{-1}.$$

This is because $f_s(r) = \gamma_{qr}(u_s(q) - u_s(r))$.

IV. Inverse problem without a diode

The matrix F is a collection of vectors f_i, which are the outflows at each boundary node due to a current of 1 at node i. The diagonal entries of A^{-1} are the potentials at each interior node due to a current at that node. In the inverse problem without a diode we are given the matrix F and the diagonal entries of A^{-1}. We will begin recovering the conductors in the network at the boundaries and we will work our way in to the middle.

To recover the boundary conductors, recall definition 3.1 . To solve for γ_{qr}, where r is a boundary node and q is its interior node, we use the information $f_s(r)$, $u_s(q)$, and $u_s(r) = 0$ in the following equation:

$$f_s(r) = \gamma_{qr}(u_s(q) - u_s(r))$$

Therefore,

$$\gamma_{qr} = f_s(r) / u_s(q).$$

The above procedure holds for all boundary conductors.

• In later sections of this paper, we will input a current of 1 at one interior node and a current of α at an adjacent interior node. This will create a current flow of zero over a desired conductor. The method for calculating α is as follows:

Method 4.1

The value of $\psi_{i,j}(hk)$ is the flow from node h to node k due to a current of 1 at node i and an input current of α at node j. To calculate the flows out of the boundary nodes when using two input sources, 1 and α, we begin by constructing x, a vector of source inputs. The entry in row i is 1 when we input a current of 1 at node i. The entry in row j is α when we are inputting a current of α at node j. All other entries are 0. The vector x will look like

$$x = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ \vdots \\ \alpha \\ \vdots \\ 0 \end{bmatrix} \begin{array}{l} \\ \\ \Leftarrow \text{ ith row} \\ \\ \\ \\ \Leftarrow \text{ jth row} \\ \\ \\ \end{array}$$

Then, we multiply x by A^{-1} to get y where y is a vector of potentials at each interior node due to input currents of 1 at node i and α at node j. So,

$$y = A^{-1}x.$$

Then for each interior node h in the 1-deep ring with neighboring boundary node k, we use the following equation to calculate $g_{i,j}(k)$

$$\psi_{i,j}(k) = \gamma_{hk}[(u_i(h) + \alpha u_j(h)) - (u_i(k) + \alpha u_j(k))].$$

Since the potential at any boundary node is 0, this equation can be simplified to

$$\psi_{i,j}(k) = \gamma_{hk}(u_i(h) + \alpha u_j(h)).$$

- The above argument can be used not only to calculate the flows out of boundary nodes, but also for calculating the flows across interior conductors. ■

Now we will recover the conductors on the 1-deep ring. Begin by recovering the conductors closest to the corners. Referring to figure 4.2 below, we will begin by recovering γ_{hp}.

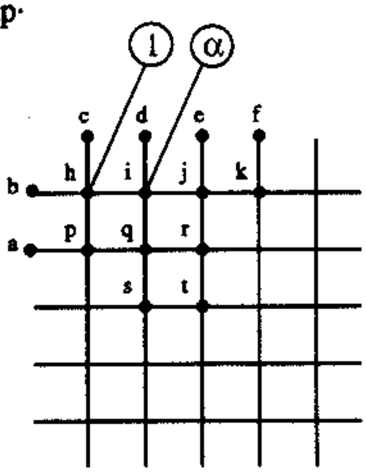
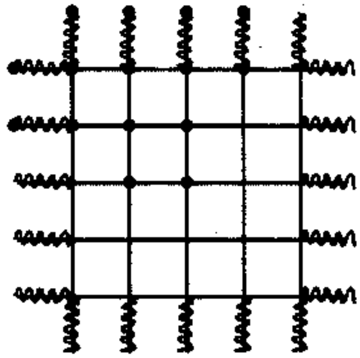


Figure 4.2



Currently Recovered Conductors

We need to find $u_h(p)$ and $u_h(i)$. We will also need to find $u_i(h)$, which will in fact be the same as $u_h(i)$, recalling that A^{-1} is symmetric.

We can calculate these potentials by dividing the current out of the adjacent boundary node by the value of the conductor between the interior node and the boundary node.

Using the information $f_h(b)$, $u_h(b) = 0$ and γ_{hb}, we will solve for $u_h(p)$ in the equation

$$f_h(b) = \gamma_{hb}(u_h(p) - u_h(b))$$

Similarly, we can solve for $u_h(i)$.

To solve for γ_{hp} we must use two input currents, as described in method 4.1. Let the two source nodes be node h and node i (figure 4.2). Let the input current at node h be 1 and at node i be some variable α. The object is to find the value of α so that there is zero current flow between h and i. This occurs when the potentials of node h and node i are equal, as calculated in the equation

$$u_h(h) + \alpha u_i(h) = u_h(i) + \alpha u_i(i).$$

By Kirchhoff's Law the current flowing into a node is equal to the current flowing out of a node. Recalling the notation used in definition 3.2 (describing current flow between interior nodes), it is true for node h that

$$1 = g_{h,i}(b) + g_{h,i}(c) + \psi_{h,i}(hp) + \psi_{h,i}(hi)$$
$$1 = g_{h,i}(b) + g_{h,i}(c) + \psi_{h,i}(hp)\,.$$

Solving for $f_{h,i}(hp)$, we have

$$\psi_{h,i}(hp) = 1 - (g_{h,i}(b) + g_{h,i}(c)).$$

Now we can solve for γ_{hp} from the equation

$$\psi_{h,i}(hp) = \gamma_{hp}[(u_h(h) + \alpha u_i(h)) - (u_h(p) + \alpha u_i(p))].$$

Similarly, we can solve for γ_{hi} by letting the input current at h be 1 and the input current at p be a different α.

• From here we will continue using the method of inputting 1 and α into neighboring nodes, as described in method 4.1. If we know two of the currents out of an interior node, then we can create a current of zero across one of the node's remaining conductors. Then we can solve for the fourth current and ultimately, that conductor. We have shown how to do this for interior nodes on the 1-deep ring. We will use the same process for interior nodes on the deeper rings. However, it is a lengthier process to calculate the potentials of these nodes.

Now, we will solve for the conductors on the 2-deep ring. This procedure will also recover the conductors between the 1-deep and 2-deep rings. Refer to figure 4.3 below.

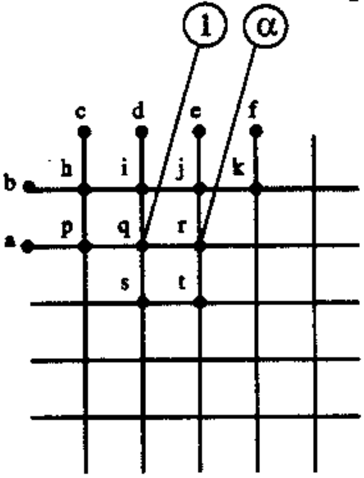
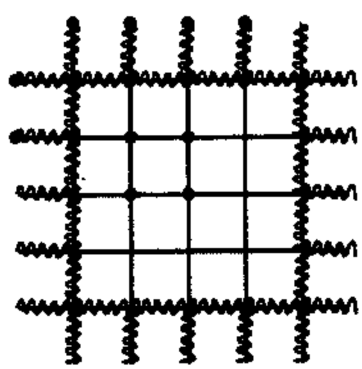


Figure 4.3



Currently Recovered Conductors

First, we need to calculate $u_q(i)$, $u_q(j)$, and $u_q(k)$.

$$u_q(i) = f_q(d) / \gamma_{id}$$
$$u_q(j) = f_q(e) / \gamma_{je}$$
$$u_q(k) = f_q(f) / \gamma_{kf}$$

From these potentials we can calculate $\phi_q(ji)$, $f_q(e)$ and $\phi_q(jk)$ using the following equations.

$$\phi_q(ji) = \gamma_{ji}(u_q(j) - u_q(i))$$
$$f_q(e) = \gamma_{je}(u_q(j))$$
$$\phi_q(jk) = \gamma_{jk}(u_q(j) - u_q(k))$$

Now that we have three of the four currents out of node j, we can calculate the current flow between node j and node r by Kirchhoff's Law using the equation below.

$$0 = \phi_q(ji) + f_q(e) + \phi_q(jk) + \phi_q(jr)$$

Solve this for $\phi_q(jr)$. Then we use this result in the following equation and solve for $u_q(r)$.

$$\phi_q(jr) = \gamma_{jr}(u_q(j) - u_q(r)).$$

Recall that A^{-1} is symmetric. Therefore, by finding $u_q(r)$ we have also found $u_r(q)$. Since we know these potentials we can now use the method of inputting 1 at node q and α at node r to recover the conductors in the 2-deep ring.

• The above process (for calculating the potentials and current flows at interior nodes) can be extended up to an n-deep ring. Ultimately, all of the conductors for the network will be recovered.

V. Preliminaries for a network with a diode

Definition 5.1. A *diode* is a nonlinear electrical device which allows current to flow through it in only one direction. The *diode* allows current to flow between its base and tip if the potential at its base is greater than the potential at its tip. In the case where the potential at the base is less than the potential at the tip, the diode becomes an open circuit, and the current flow is 0.

Remark 5.2. •Assume now that there is exactly one diode in the network, and that for each experiment we perform that the potentials on the boundary are 0. We will also assume that the diode is between interior nodes x and y and conducts with γ_{xy} when the diode is on, and 0 when the diode is off. For each interior node in the network, the term ON will be used to represent normal current flow over the conductor with the diode (diode does not block current flow). The term OFF will be used to represent zero (0) current flow over the conductor with the diode (diode blocks current flow). The term

NEUTRAL will be used to represent a node which creates a current flow of zero (0) over the conductor with the diode (base and tip potentials are equal). Since a neutral node creates a current flow of zero, all neutral nodes will be included in the set of OFF nodes.

Definition 5.3 Let P be the set of all interior nodes that turn the diode on, and Q be the set of all nodes that turn the diode off. It is clear that $P \cup Q$ is the set of all nodes.

- In order to simulate a network Γ with a diode, we will now create two auxiliary networks which we will call Γ_1 and Γ_0. Γ_1 will be the network Γ with the diode removed and the conductor γ_{xy} in its place. Γ_0 will be the network Γ with the diode and its conductor removed. (see Figures below)

Remark 5.4 • We will get A_1^{-1} and A_0^{-1} in the same manner that we obtained A^{-1} for the network Ω in the forward problem without a diode. We also know that since neither Γ_1 nor Γ_0 have a diode. This implies that A_1 and A_0 are symmetric. Therefore A_1^{-1} and A_0^{-1} are symmetric. We will also define G to be the matrix of potentials for the network Γ.

Definition 5.5 Let r be an boundary node, and let p be any interior node. When we are in Γ. Let $f_p(r)$ denote the flows out of r due to a current of 1 at node p When we are in Γ_1 we will denote the flows out of the boundary to be $f_p^{(1)}(r)$. When we are in Γ_0 we will denote the outflows to be $f_p^{(0)}(r)$. And similarly if we input a source of -1 at p the flows would be $f_{-p}(r)$, $f_{-p}^{(1)}(r)$, and $f_{-p}^{(0)}(r)$.

Remark 5.6. • In Γ we will now assume that γ_{xy} is the conductor with the diode. We will also assume that the diode allows flow from x to y, and creates zero flow from y to x.

- In Γ_1, γ_{xy} will be in the same position as it is in Γ, except there will be a flow from x to y, and a flow from y to x.
- In Γ_0, γ_{xy} will be removed, and there will be zero flow from x to y and y to x.

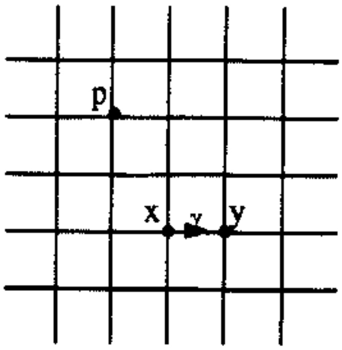


Figure 5.1

Γ- Network

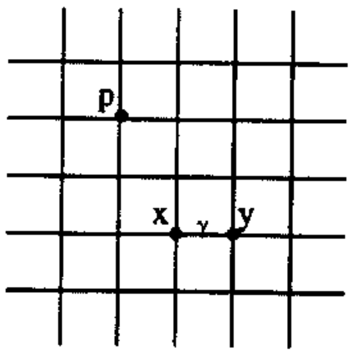


Figure 5.2

Γ_1- Network

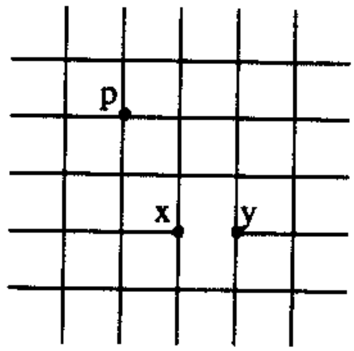


Figure 5.3

Γ_0- Network

Definition 5.7. Let u_p be the vector of potentials due to a current of 1 at node p in Γ_1. Let v_p be the vector of potentials due to a current of 1 at node p in Γ_0. Let w_p be the vector of potentials due to a current of 1 at node p in Γ .

Theorem 5.8. Let Γ be any rectangular network with a diode. Then there is at least one interior node in the network that turns the diode on, and one interior node that turns the diode off.

Proof:
Let node $p \in \mathrm{int}(\Omega_0)$. Then by the maximum principle, we know that then $u_x(x) > u_x(p)$ for any $p \in \mathrm{int}(\Omega_0)$. Therefore $u_x(x) > u_x(y)$. Hence node x turns the diode on. Similarly, $u_y(y) > u_y(p)$ for all $p \in \mathrm{int}(\Omega_0)$. Therefore $u_y(y) > u_y(x)$. Hence node y turns the diode off. Therefore for rectangular network with a diode there is at least one node that turns it on and one that turns it off. ■

Theorem 5.9. (Refer to Figures 5.2 & 5.3) Let u_p be the potentials due to a current of 1 at node p in Γ_1, and v_p be the potentials due to a current of 1 at node p in Γ_1.
In Γ_1 Let β be the flow from x to y due to a current of 1 at node p.
Let δ be the flow from x to y due to a current of 1 at x. Let ε be the flow from x to y due to a current of 1 at y. Let $\omega = \beta/(1-\delta-\varepsilon)$. Then $v_p = u_p + \omega u_x - \omega u_y$.

Proof:
Now in the Γ_0, we want to calculate the currents at x.
At node x, the the currents due to potential u_p total $-\beta$
At node x, the the currents due to potential u_x total $1 - \delta$
At node x, the the currents due to potential u_y total $-\varepsilon$

Then,

$$\begin{aligned} u_p + \omega u_x - \omega u_y &= u_p + (\beta/(1-\delta-\varepsilon))u_x - (\beta/(1-\delta-\varepsilon))u_y \\ &= -\beta + (\beta/(1-\delta-\varepsilon))(1-\delta) - (\beta/(1-\delta-\varepsilon))(-\varepsilon) \\ &= (-\beta + \delta\beta + \varepsilon\beta + \beta - \beta\delta - \beta\varepsilon)/(1-\beta-\gamma) \\ &= 0 / (1-\beta-\gamma) \\ &= 0 \end{aligned}$$

Hence $u_p + \omega u_x - \omega u_y$ satisfies Kirchhoff's Law at all nodes in the Γ_0.
Therefore $v_p = u_p + \omega u_x - \omega u_y$. ■

Theorem 5.10. Let p be a node that turns the diode on, and let q be a node that turns the diode off. Then $w_p = u_p$ and $w_q = v_q$.

Proof:
We have already assumed that Γ_1 will have the appearance that all of the interior nodes turn the diode on. We have also assumed that Γ_0 will have the appearance that all of the interior nodes turn the diode off. Therefore in the forward problem if a node p turns the diode on in Γ then $w_p = u_p$. If node q turns the diode off then $w_q = v_q$. ■

Theorem 5.11. (Refer to Figures 5.2 & 5.3) Let u_p be the vector of potentials due to a current of 1 at node p in Γ_1. Let v_p be the vector of potentials due to a current of 1 at node p in Γ_0. Then for all nodes p in the interior of Γ_1 (and Γ_0), $v_p(p) \geq u_p(p)$.

Proof:
We know that $v_p = u_p + \omega u_x - \omega u_y$.

Case 1.

Let p be an interior node which turns the diode on.
Then $u_p(x) > u_p(y)$
Therefore $u_x(p) > u_y(p)$

We then know that $\beta > 0$, because $u_p(x) > u_p(y)$.
Now, if we put a current of 1 at x and -1 at y, we know that some of the current applied at x will flow to each neighbor of x. Hence the flow across γ_{xy} cannot equal 1. Thus we can see that $\delta + \varepsilon < 1$.
Therefore we know that $\omega > 0$.
Hence we know that $\omega u_x(p) - \omega u_y(p) > 0$.
Therefore $v_p(p) > u_p(p)$ for all nodes p that turn the diode on.

Case 2.

Let p be an interior node that turns the diode off.
Then $u_p(x) \leq u_p(y)$
Therefore $u_x(p) \leq u_y(p)$

We then know that $\beta \leq 0$, because $u_p(x) \leq u_p(y)$.
Now, if we put a current of 1 at x and -1 at y, we can see that $\delta + \varepsilon < 1$.
Therefore we know that $\omega \leq 0$.
Hence we know that $\omega u_x(p) - \omega u_y(p) \leq 0$.
Therefore $v_p(p) \geq u_p(p)$ for all nodes p that turn the diode off.

Therefore $v_p(p) \geq u_p(p)$, for all nodes p in the interior of the Γ_1 (and Γ_0). ■

Recall that $u_{-p}(j)$ is the potential at node j due to a current of -1 at node p.

Theorem 5.12. Let node p be an interior node in a network without a diode. Let x and y be any other neighboring interior nodes, such that $u_p(r) > u_p(s)$.
Then $u_{-p}(r) < u_{-p}(s)$.

Proof:
We know that $u_p(r) > u_p(s)$.
Then if we input a current of -1 at node p we can simulate what will happen by multiplying u_p by -1. In this case $u_p = -u_{-p}$. Therefore $u_p(r) = -u_{-p}(r)$.
Since $u_p(r) > u_p(s)$ it follows that $-u_{-p}(r) > -u_{-p}(s)$. Which implies that $u_{-p}(r) < u_{-p}(s)$.
Therefore $u_{-p}(r) < u_{-p}(s)$. ■

Theorem 5.13. Let p be a node that turns the diode on. Then $-w_{-p}(p) = v_p(p)$.

Proof:
Since node p turns the diode on, $u_p(x) > u_p(y)$ and $w_p(x) > w_p(y)$. Then if we put a current of -1 in at node p, by theorem 5.12 we know that $u_{-p}(x) < u_{-p}(y)$. This implies that putting a current of -1 in at node p will turn the diode off. This implies that $-w_{-p} = v_p$. Therefore $-w_{-p}(p) = v_p(p)$. ■

Theorem 5.14. Let q be a node that turns the diode off. Then $-w_{-q}(q) = u_q(q)$.

Proof:
The proof follows directly from theorem 5.13. ■

Theorem 5.15. Let p be a node that turns the diode on. Then $w_p(p) < -w_{-p}(p)$.

Proof:
We know from theorem 5.11 that if node p turns the diode on then $u_p(p) < v_p(p)$. We also know from theorems 5.10 and 5.13 that $w_p(p) = u_p(p)$ and $-w_{-p}(p) = v_p(p)$, therefore $w_p(p) < -w_{-p}(p)$. ■

Theorem 5.16. Let q be a node that turns the diode off. Then $w_q(q) > -w_{-q}(q)$.

Proof:
It is clear that by a similar argument to theorem 5.15, it can be shown that
$w_q(q) \geq -w_{-q}(q)$. ■

Recall that f_p is the vector of outflows due to a current of 1 at node p. Also recall that f_{-p} is the vector of outflows due to a current of -1 at node p.

Theorem 5.17. Let p be any node that turns the diode on and q any node that turns the diode off. Let i be any boundary node. Then the following will occur:
1. $f_p(i) = f_p^{(1)}(i)$.
2. $-f_{-q}(i) = f_q^{(1)}(i)$.
3. $f_q(i) = f_q^{(0)}(i)$.
4. $-f_{-p}(i) = f_p^{(0)}(i)$.

Proof:
We know that if node p turns the diode on then $w_p = u_p$. This implies that $f_p = w_p C^T = u_p C^T$. Therefore $f_p(i) = f_p^{(1)}(i)$. We also know, by theorem 5.14, that if node q turns the diode off $-w_{-q} = u_q$. Since $f_{-q} = w_{-q} C^T$, we can conclude that $-f_{-q} = -w_{-q} C^T = u_q C^T$. Therefore $-f_{-q}(i) = f_q^{(1)}(i)$ in Γ_γ. By a similar process we can also prove #3 and #4. ■

V. Forward problem with a diode

Now we are ready to simulate the forward problem with a diode.

Step 0. In this step we create our auxiliary networks Γ_1 and Γ_0. We create them as they are described in Remark 5.6.

Recall Remark 5.4. for the notation A_1^{-1}, A_0^{-1} and G.

Step 1. In step 1 we will use the network Γ_1 to determine which nodes will turn the diode on, and which nodes will turn the diode off. We do this by calculating A_1^{-1}, just as we did in the forward problem without a diode. Once A_1^{-1} has been calculated we will compare $u_p(x)$ and $u_p(y)$ for all p in int(Γ). With this calculation we know that if $u_p(x) > u_p(y)$ then node p turns the diode on, otherwise node p turns the diode off.

Step 2. Now we would like to use A_1^{-1} and A_0^{-1} to calculate G, where each column of G comes from a column in either A_1^{-1} or A_0^{-1}. Since we have A_1^{-1} and A_0^{-1} we can begin to create G. We already know which nodes turn the diode on and which nodes turn the diode off. Let p be any interior node in the network; If p turns the diode on then we will let $w_p = u_p$ (the corresponding column from A_1^{-1}). If node p turns the diode off we will let $w_p = v_p$ (the corresponding column from A_0^{-1}). Hence, G is created from A_1^{-1} and A_0^{-1}.

Step 3. We now simply want to calculate the outflows from Γ. We can now do this by calculating $-F = GC^T$.

VI. Inverse problem with a diode

We are now ready to begin recovering the network Γ.

Step 0. In this step we will begin solving the inverse problem by applying currents of 1 and -1 at each interior node. Now recall that $w_{-p}(p)$ is the potential at node p due to a current of -1 at node p, and $f_{-p}(i)$ is the flow out boundary node i due to current of -1 at node p. From this we step will be able to get the $w_p(p)$ and $w_{-p}(p)$ at all interior nodes p in Γ. We will also be able to get $f_p(i)$ and $f_{-p}(i)$, for all $p \in \text{int}(\Gamma)$ and $i \in \partial(\Gamma)$.

Step 1. At this point we will separate the nodes into two groups, those nodes which turn the diode on and those nodes which turn the diode off . From theorem 5.15 we can see that if $w_p(p) < -w_{-p}(p)$ then node p turns the diode on. From theorem 5.16 we know that if $w_p(p) > -w_{-p}(p)$ then node p turns the diode off. By theorem 5.8 we know that at least one node must turn the diode on and one node must turn the diode off.

Step 2. Now we will show how to recover both the diagonal entries of A_1^{-1} and A_0^{-1}. By theorem 5.16 we know that $-u_{-p}(p) = v_p(p)$ and $-v_{-p}(p) = u_p(p)$. We also know that if node p turns the diode on then $w_p(p) = u_p(p)$ and if it turns the diode off then $w_p(p) = v_p(p)$. We also know that if node p turns the diode on then $-w_{-p}(p) = v_p(p)$ and if it turns the diode off then $-w_{-p}(p) = u_p(p)$. Recall Definition 5.3. Now for all nodes p that turn the diode on and all nodes q that turn the diode off, $w_p(p) = u_p(p)$ and $-w_{-q}(q) = u_q(q)$. Therefore we have recovered all of the diagonal entries of A_1^{-1}. Similarly, we can recover all of the diagonal entries of A_0^{-1}.

Step 3. In step 3 we will show how to recover all of the outflows for Γ_1 and Γ_0. Recall theorem 5.17 and definition 5.3. Now for $p \in P$ and $q \in Q$ we know that $f_p(i)$ in Γ will equal $f_p(i)$ in Γ_1 and $-f_{-q}(i)$ in Γ will equal $f_q(i)$ in Γ_1. We also know that $f_q(i)$ in Γ will equal $f_q(i)$ in Γ_0 and $-f_{-p}(i)$ in Γ will equal $f_p(i)$ in Γ_0. Therefore we now have all of the outflows for Γ_1 and Γ_0.

Recall in remark 5.5 we know that A_1^{-1} is symmetric, A_0^{-1} is symmetric but that G is not symmetric.

Step 4. Because our algorithm for recovering a network without a diode depends on the symmetry of A^{-1}, we cannot use it directly on G. So, we must use the auxiliary networks in order to use our method.

Now we will show how to recover all of the resistors for Γ_1 and Γ_0. Recall that A_1^{-1} and A_0^{-1} are auxiliary networks without diodes. We have just shown in steps two and three that we have all of the diagonal entries of A_1^{-1} and A_0^{-1}. We also have all of the outflows for Γ_1 and for Γ_0. We also know that A_1^{-1} and A_0^{-1} are symmetric. First we will recover Γ_1 using the same method we used to recover a network without a diode from Section IV. Then we will recover Γ_0 using the same method from Section IV.

Step 5. In this section we will show how to recover Γ. From step 4 we now have Γ_1 and Γ_0. Γ_1 will give us all of the conductors of Γ. Γ_0 will give us all of same conductors of Γ except there will be a zero where the diode is. Therefore we know all of the conductors of Γ and the location of the diode. We can also find the direction of the diode. Of the two nodes connected to the conductor with the diode, we know which node turns the diode on and which node turns it off. We also know that the direction of the diode, from base to tip, goes from the node that is on to the node that is off. Hence we have recovered all of Γ.

VI. Conclusions

1. It would be interesting to consider a network with more than one diode. This investigation might discover further properties of networks, or even alter the theorems presented in this paper. Undoubtedly, more than one diode in a network will create even greater complexity for the inverse problem.

2. The degree of accuracy involved in the recovering of a network with a diode is also of interest. Examining the accuracy of the recovered conductors, as the network gets larger and larger, would make an excellent project.

3. The algorithm for locating a diode in a network (as defined in this paper) is currently being developed into a computer program. Once complete, this program will allow for the further investigation of network properties, as suggested above.