Square Conductor Networks
of the Solar Cross Type

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Abstract

Methods for recovering the conductances of solar cross networks from their Dirichlet-to-Neumann maps are discussed. Three algorithms are given for the solution of this problem.

1 Terminology: what is a λ-matrix?

As in [2], consider a network Ω_β of conductors in square-lattice form (Figures 1 and 2).

Figure 1: general, Ω_β

Figure 2: square, $\Omega_{4;\beta}$

The conductances $\beta = \{\beta_i\}$ give rise to an operator Λ (more precisely, Λ_β), the Dirichlet-to-Neumann operator for the network Ω_β; if $\vec{\varphi}$ is a vector representing the voltage potential at each boundary node of Ω_β, the vector of currents $\Lambda_\beta\vec{\varphi}$ represents the current flow at each boundary node, respectively. The matrix representations of these Dirichlet-to-Neumann maps are known as λ-matrices. When the network Ω is an $n \times n$ square network, as shown above, it will be called $\Omega_{n;\beta}$, and the Dirichlet-to-Neumann map $\Lambda_{n;\beta}$, respectively.

In this paper I am primarily concerned with the cross network, a refinement of the ordinary square-lattice conductor network, that differs only in that it is composed of elements of four conductors of equal conductivity, arranged in a Greek or solar cross. The Dirichlet-to-Neumann map for such a network Ω^+_α will be called Λ^+_α, or simply Λ^+, and the set of such maps, a subset of that for ordinary networks, will be called the set of λ^+-matrices.

2 Motivation: why solar crosses?

Unlike the ordinary square-lattice network, the cross network lends an easy interpretation to the voltage potential at the exterior nodes, and of the conductances of its conductors, as an approximation to the case of continuously varying conductance in a two-dimensional domain: the potential at and current through a node correspond to the total[1] potential of, and flux through, the face of an approximating block of constant conductivity, and the conductivity α to twice the constant conductivity density γ. Another type of element with the same properties, and indeed equivalent to the cross as a module, is that of a flattened tetrahedron whose edges all have the same conductivity, i.e., $\frac{1}{2}\alpha_{cross} = \gamma = 2\alpha_{tetra}$. The lack of a central node, however, makes them significantly more difficult to work with.

The cross network also gives special properties to the Λ^+ map, as will be discussed in the following sections.

From this point I shall discuss only *square* networks.

3 Forward and Inverse problems: $\alpha \rightleftharpoons \Lambda^+_\alpha$

3.1 Ordinary Derivations: $\alpha \rightleftharpoons \beta \rightleftharpoons \Lambda_\beta \rightleftharpoons \Lambda^+_\alpha$

The forward problem, that of determining the Dirichlet-to-Neumann map Λ from the known conductances $\{\alpha_i\}_{i=1}^{n^2}$, proceeds directly, using techniques detailed in [2], [3], [4], and with the knowledge of the conductance formula for

[1] $\int_{edge} u\, d\lambda$ and $\int_{edge} \frac{\partial u}{\partial \vec{n}}\, d\lambda$, respectively, where u is the potential function.

conductors in series[2],

$$\alpha_{equivalent} = \frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2}. \tag{1}$$

Likewise the inverse problem, once solved using the methods described in [2] for the square-lattice, can be further solved for all conductances α_i by "peeling off" crosses from the boundary: the conductance of each boundary conductor is equal to that of every other conductance in its cross, and known crosses can be removed, revealing a new boundary. A preferable[3] method, however, is to use the relationships generated by the fact that the network is of solar cross type to generate a different algorithm, as in section 3.3.

3.2 Relations: k-corners and opposing sides

3.2.1 Corners

In [3] it is shown that if zero potentials and zero currents are imposed along the North and West faces of a square network (Figure 3), the resulting potentials in all nodes north and west of the diagonal must be zero; and this diagonal demarcation line may be moved NW by removing the restrictions at the "end" nodes, those furthest south and east respectively, or SE by imposing the same condition of zero voltage and zero current at the northern nodes of the east face and the western nodes of the south face.

Figure 3: a 4-corner relationship for $\Omega_{4;}$

[2] more commonly known for resistors in series, $R_{equiv} = R_1 + R_2$; but conductance is the inverse of resistance, so $\frac{1}{\alpha_{equiv}} = \frac{1}{\alpha_1} + \frac{1}{\alpha_2}$.

[3] in the sense of fewer operations

Known voltages placed at the other South-face boundary nodes may determine voltages and currents through the rest of the network, including the other (unrestricted) boundary nodes; in this manner, a "k-corner relationship", a linear system of the form

$$\sum_{i=0}^{2k-1} x_i \lambda_{m,j+i} = 0, \qquad \text{for all } m \notin \{j, j+1, ..., j+2k-1\}. \tag{2}$$

is produced from Λ_β and k, where the indices $j+k$ and $j+k-1$ correspond to the two "corner" nodes, and j and $j+2k-1$ to the "end" nodes.[4]

In [3] it is shown that (2) is over-determined[5] when $k < n$, and under-determined when $k > n$. In the case of the over-determined systems, it is proved in [3] that although the system (2) is over-determined, it has a solution $\vec{x} = (x_0, ..., x_{2k-1})$ dependent only on $x_0, ..., x_{k-1}$.

The 1-corner relationship for cross networks is particularly simple, stating that

$$\lambda_{m,j} - \lambda_{m,j+1} = 0 \qquad \text{for all } m \notin \{j, j+1\} \tag{3}$$

and

$$\lambda_{j,j} - \lambda_{j+1,j+1} = 0 \tag{4}$$

where j is one of n, $2n$, $3n$, $4n$.

Figure 4: the parallel-sides relationship for a 3×3 network

[4] Of course, consider all indices as residues modulo $4n$, e.g. $4n+1$ and 1 are equivalent as indices. Also note that for every k there are exactly *four* possible choices of j: $j = n-k+1, 2n-k+1, 3n-k+1, 4n-k+1$.

[5] $x_k, ..., x_{2k-1}$ in terms of $x_0, ..., x_{k-1}$.

3.2.2 Remark

In the case of cross networks, this method discovers a relationship between $\Lambda^+_{n;\alpha}$ and $\Lambda^+_{n-1;\alpha}|_{\Omega^+_{n-1}}$: when $k = n$, the same coefficients $\{x_i\}$ will solve the two equations

$$\sum_{i=0}^{2k-1} x_i \lambda^+_{n-1;m,j+i} = 0, \qquad \text{for all } m \notin \{j, j+1, ..., j+2k-1\} \tag{5}$$

and

$$\sum_{i=0}^{2k-1} \chi(i) x_i \lambda^+_{n;m,j+i} = 0, \qquad \text{for all } m \notin \{j, j+1, ..., j+2k-1\}, \tag{6}$$

where $\quad \chi(i) = \begin{cases} +1 & \text{if } 0 < i < 2k-1; \\ -1 & \text{if } i = 0 \text{ or } i = 2k-1 \end{cases}.$

3.2.3 Cone relations

In the case that n is odd, the following relationship is also possible. Specifying voltage and current zero at all nodes on the north and south faces, as in Figure 4, the system

$$\sum_{i=1}^{n} x_i \lambda^+_{m,i} + \sum_{i=2n+1}^{3n} x_{i-n} \lambda^+_{m,i} = 0 \qquad \text{for all } m \in \{n+1, ..., 2n\} \cup \{3n+1, ..., 4n\} \tag{7}$$

uniquely determines $\vec{x}$ as a function of x_1. The same relation can be found with respect to two adjacent sides, as a special case of (2).

In any case (n even or odd), a relation similar to (7) holds: given any interior node that is the center of a cross, it is possible to force the potential to be zero at that node and all nodes whose heading[6] θ, relative to the node in question, satisfies $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$ or $\frac{3}{4}\pi \leq \theta \leq \frac{5}{4}\pi$, by solving the (over-determined) system

$$\sum_{i=2n-y-z}^{2n-y+z} x_{i-2n+y-z} \lambda^+_{m,i} + \sum_{i=2n+y+z}^{4n-y-z} x_{i-2y} \lambda^+_{m,i} = 0, \tag{8}$$

$$\text{for all } m \notin \{2n-y-z, ..., 2n-y+z\} \cup \{2n+y+z, ..., 4n+y-z\}$$

for $\vec{x}$ in terms of x_0, where (y, z) are the Cartesian coördinates of the cross element in question, considering the south-western-most element as $(0, 0)$. This is exactly relation (7) when $y = z = \frac{n-1}{2}$.

[6] where due east is considered to be heading zero, and proceeding counter-clockwise to 2π, which is again due east.

3.3 Specialized derivations: $\alpha \leftarrow \Lambda_\alpha^+$

The algorithm to find α proceeding inward from the corners is described in detail in [2]; the only additions are as follows:

- The corner conductances are immediately evident, for example $\alpha_{SE} = \lambda_{n,n} - \lambda_{n,n+1}$, from the 1-corner relationship.
- After finding the conductances for each ladder, the known "half" of the conductance associated with a cross of known conductance must be "subtracted" using (1) to find the next diagonal of crosses. Since only one leg of a cross need be computed, only half of the information gained in the solution to the general inverse problem need be employed; or, alternatively, the redundant information can be used to give greater accuracy.

Using the corner method, n^2 variables $x_{j;k;i}$ must be determined, and if all of the corner relationships are used, the system can be redundantly over-solved $2 + \frac{2}{n}$ times.

If the conductance of a specific element is all that is desired, relation (8) may prove useful. The conductances of boundary elements can be solved for directly, setting $x_0 = 1$; α_*, the conductance in question, is given by

$$\alpha_* = \sum_{i=1}^{4n} \varphi_i \lambda_{m,i}^+ \tag{9}$$

where $\vec{\varphi}$ is the boundary potential vector obtained from (8).

Figure 5: a cone relationship

Once three adjacent boundary crosses are known, relation (8) can again be used to find the conductance of the element neighboring the centermost of the three elements; and in general, if the conductances of all crosses forming a wedge, the tip of which is the "element in question", are known, (8) can be used to determine its conductance.

To solve the entire inverse problem in this fashion would require solving for $4\sum_{i=1}^{n/2}(n-i)(2n-2i+1)$ variables; it is apparent that this method is useful primarily for determining the conductance of single elements at or near the edge of a large network.

3.3.1 Isolation

The method of "resistor isolation" given in [4] can also be applied to cross networks, although unlike general square-lattice networks, a path of elements with known conductances must join the element in question with the boundary if its conductance is to be determined.

Figure 6: Landrum's "resistor isolation"

In the general case, the method proceeds as follows: the conductor in question, which I shall call I, has two nodes: a "top" node and a "bottom" node. To all boundary nodes whose heading, relative to the "top" node satisfies $\frac{3\pi}{4} \leq \theta \leq \frac{5\pi}{4}$, assign the potential $+1$, and to all boundary nodes whose heading, relative to the "bottom" node, satisfies $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$, potential 0, and to all of these boundary nodes assign current 0. This forces two wedges of constant potential, as shown in Figure 6. The system

$$\sum_{i=2n-y-z}^{2n} x_{2n-y-z+i}\lambda_{m,i} + \sum_{i=2n-y+z+2}^{2n+x+y+2} \lambda_{m,i} + \sum_{i=3n}^{4n+y-z} x_{i-2z-2}\lambda_{m,i} = 0, \quad (10)$$

for all $m \in \{y-z+1, ..., 2n-y-z-1\} \cup \{2n-y+z+2, ..., 2n+y+z+2\}$,

where (y, z) are the Cartesian coördinates of the bottom of I, with $(0, 0)$ designating the south-western-most interior node, can be solved for $\vec{x}$, and hence for $\vec{\varphi}$, the $4n$-dimensional vector of potentials around the boundary. In [4] it is shown that the current through I, which is equal to the total flux through the "top" of the network, i.e.

$$\sum_{i=2n+z+1}^{5n-z} \lambda_{i,j}\varphi_i \quad (11)$$

is also equal to the conductance of I, by Ohm's law.

Additional conductances are gained where the wedges of potential 0 and 1 join to the nodes whose potentials were determined by (10); since the potential of the nodes in the wedge is fixed, and the potential at and current through the boundary node is known, Ohm's law will reveal the conductance of the joining conductor. At least two conductances can be gained in such a fashion, raising the total for this procedure to at least three conductances–although, as we are warned in [4], accuracy rapidly drops as (y, z) is removed from $(\frac{n-1}{2}, \frac{n-1}{2})$.

Unfortunately, however, returning to the cross network, this information is not sufficient to recover any cross that does not border on another element whose conductance is known.

4 Characterization: Which Λ's are Λ^+'s?

Obviously, not all square-lattice networks are of solar cross type; but every cross network generates a square-lattice network from which it is recoverable. The question arises, then, how to tell the *legal* cross networks from *illegal* or ordinary square-lattice networks. If the conductances are known, boundary crosses need only be peeled off until none remain (legal) or a contradiction is reached (illegal).

Alternatively, the network can be subjected to the k-corner relationships stated in section 3.2.1, in a partial solution of the inverse problem; if any contradictions arise, e.g. if $\lambda_{n;m,n} - \lambda_{n;m,n+1} \neq 0$ for any $m \notin \{n, n+1\}$, this Λ is not a legal λ^+-matrix.

Works Cited

[1] Calderon, A. P., On an Inverse Boundary Value Problem.
[2] Curtis, Edward B. and Morrow, James A.,
Determining the Resistors in a Network. SIAM J. Appl. Math, Vol. 50, No. 3, pp. 918-930, June 1990.
[3] Curtis, Edward B. and Morrow, James A.,
The Dirichlet to Neumann Map for a Resistor Network. SIAM J. Appl. Math, Vol. 51, No. 4, pp. 1011-1029, August 1991.
[4] Landrum, Joshua,
A Comparison of Three Algorithms for the Inverse Conductivity Problem. August 1990.