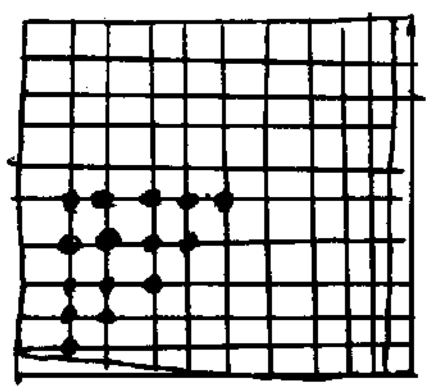
The Inverse Problem

Observations.

$U_D(j)$ resulting from any D will be some linear combination of the preceding graphs. I have only included 25 of the 81 possible points on a (10x10) grid because of symetry. (You can simply "cut" the graphs at $j=0,10,20,30$ and put them back together in an order depending on how P_1 reflects the original P_1 in the grid.) There is symetry around the set of:



n=10

D's of m=1 with the boundary of the grid. There are also equalities amongst those points relative to the j's. For instance, $r_{21}=r_{32}=r_{43}=\dots$ In actuality, there are only $(n-1)+(n-2)+(n-3)+\dots+1$ possible r_{ij}'s (minus some repeats which occur when $x_{i_1}'^2+y_{i_1}'^2 = x_{i_2}'^2+y_{i_2}'^2$). It ~~is~~ would be interesting to see how this symetry affects $u_D(j)$.

What happens for D's symetric around the triangles? Does the symetry help to solve the problem or just reduce the number of computations necessary?

Due to the geometry of the grid, each $u_{P_1}(j)$ graph has 4 local maximums and four local minimums. The minimums always occur at $j=0$, $j=n$, $j=2n$, $j=3n$.

$u_D(j)$ as a series of functions, $u_i(j)$

One approach that did not work was to ask: given the function $u_D(j)$ what combo of the $u_i(j)$ functions will add to $u_D(j)$? It is easy to visualize superposition this way, yet even if we could derive a regular, determined series of functions, $\sum f_i(x,y)$, i.e. using a describable relationship between r_{ij} and its neighbors and hence $u_i(j)$ and its neighbor $u_i(j)$'s, that equals $u_D(j)$

Illustration: 3 pts. graph total graph super-imposed. Given that D represents series

how are these $u_i(j)$ related to each other?

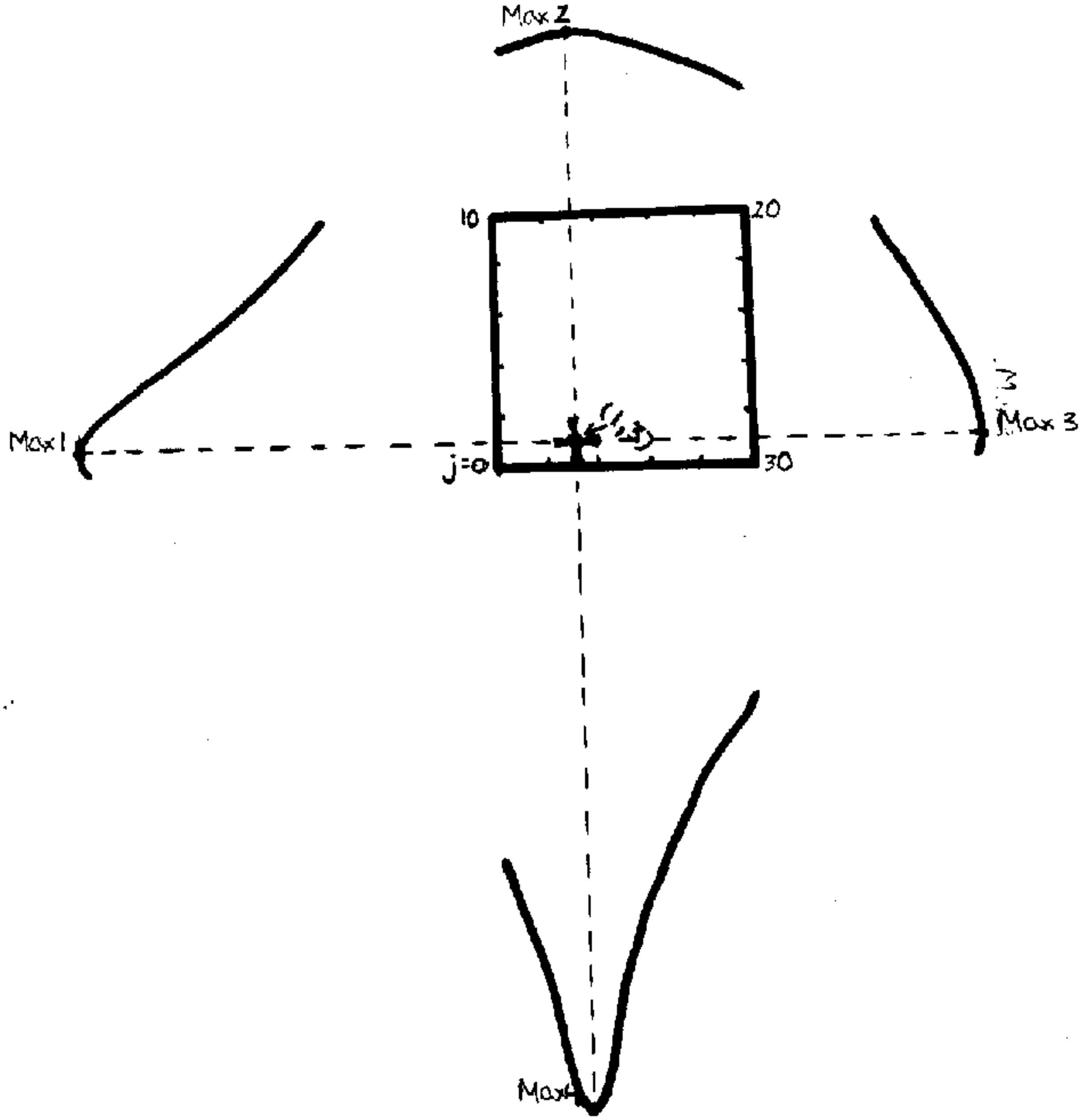
We would need greater and greater accuracy of u to find it as n increases.

Locating 1 pt on an $n \times n$ grid

If we know that $m=1$, the location of D can be found from looking at where the local maximum's of $u_0(j)$ occur. If the first local max occurs at $j = (x_a, y_a)$ and the second at $j=(x_b, y_b)$ then D occurs at $P_1 = (x_b, y_a)$. This is due to geometry and the fact that by locating the max of $u_1(j)$ we locate the min of r_{ji}.

See Illustration next page

~~Locating 2 pts.~~


Max 2
10
20
Max 1
(1,3)
Max 3
j=0
30
Max 4

Locating 2 pts.

Similarly it is not ~~too~~ difficult to locate 2 pts. given (if you are) that that's the number you have. Looking (at u) in the vicinities of the local maximums, it is possible to deduce the configuration these 2 pts. Call the local maximums max1, max2, max3, max4, according to if the occur between $j=0$ and n, n and $2n$, $2n$ and $3n$, or $3n$ and $4n$. ~~Call~~ the j's at which ~~these~~ max_i occurs j_{maxi} ~~j_{max2}, j_{max3}~~. Firstly the center of mass of the 2 pts, occurs at ~~[illegible]~~

$$\left(\frac{x_{jmax2}+x_{jmax4}}{2}, \frac{y_{jmax1}+y_{jmax3}}{2}\right).$$

If the two points lie on a line $x=c$ or $y=c$, we will find that $j_{max2} - n = 4n - j_{max4}$ or $j_{max1} = 3n - j_{max3}$, respectively. ~~If not looking at the slope of~~ The locations of P_1, P_2 are then $(j_{max2}-n, j_{max11})$ and $(j_{max2}-n, j_{max12})$ or $(j_{max21}-n, j_{max1})$ and (j_{max22}, j_{max1}), where j_{maxi1} and j_{maxi2} are the locations of the 2 equal local maximum's which occur on side i. ~~If they do not~~

Otherwise, looking at u on sides 1 and 2 will give the configuration. If $max1 - u(j_{max1}-1) < max2 - u(j_{max2}-1)$ then

$P_1 = (x_{jmax2}, y_{jmax1})$ and $P_2 = (x_{jmax2+1}, y_{jmax1-1})$

If $max - u(j_{max1}-1) > max2 - u(j_{max2}-1)$ then

$P_1 = (x_{jmax2-1}, y_{jmax1})$ and $P_2 = (x_{jmax2}, y_{jmax1+1})$

Illustration for
2 pts.

Solving for D

One approach to the inverse problem is to consider solving the equations obtained from $u_D(j)$. This seems to be only a good approach if m has been found first (otherwise you get a "variable # of variables") $u_D(j)$ for known D is a scalar function of j, with domain the integer set $\{1, \dots, 4n\}$. If we vary u over all possible D, in other words let $x_1', x_2', x_3', \dots, x_m', y_1', y_2', y_3', \dots, y_m'$ be the variables instead of j, we obtain a vector valued function $\vec{u}(\vec{x})$ which maps vectors in R^{2m} to vectors in R^{4n}. The component scalar functions of $\vec{u}(\vec{x})$ are:

fill in

With measured values of $\vec{u}(\vec{x})$ we obtain $4n$ equations with $2m$ unknowns. $\vec{u}(\vec{x})$ is, however, not a linear transformation because $\vec{u}(\alpha\vec{x}) \neq \alpha\vec{u}(\vec{x})$. My attempts to solve this set of equations lead to more and more complicated expressions, due to non-linearity.

Looking at Mass

Gauss' Law and derivatives on $\partial\Omega$

Gauss' Law tells us that the flux of the vector field through any closed surface is equal to $4\pi \times$ mass enclosed.

$$\Phi = \sum_{\text{surface}} \vec{\text{Field}}_{\text{vector}} \cdot d\vec{a}$$

$$= \sum_{\partial\Omega} -\vec{\nabla} u \cdot d\vec{a} = 4\pi(m)$$

That is the sum of the rates of change of u at the j's in the direction normal to $\partial\Omega$ will give the mass on Ω.

We are given u_j at all j and we are trying to find u_n. Is this possible?

on side:	Want to get:	from known:
$1 = \{0 \le j < 10\}$	$-u_{x_1}$	u_{y_1}
$2 = \{10 \le j < 20\}$	u_{y_2}	u_{x_2}
$3 = \{20 \le j < 30\}$	u_{x_3}	$-u_{y_3}$
$4 = \{30 \le j < 40\}$	$-u_{y_4}$	$-u_{x_4}$

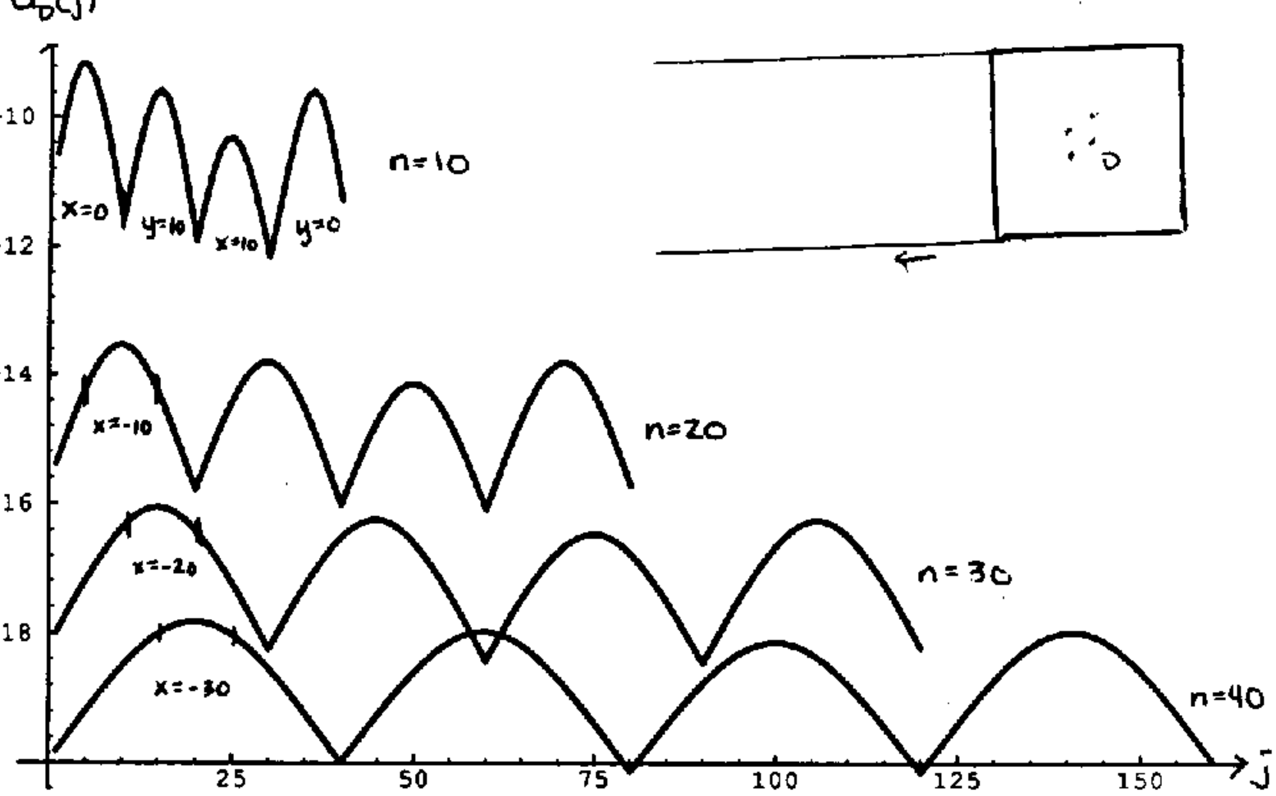
$$u_{y_1}(j) = \sum_{i=1}^{m} (y_i' - y - j)\left((x_i'-x)^2 + (y_i'-y-j)^2\right)^{-1}$$

$$-u_{x_1}(j) = -\sum_{i=1}^{m} (x_i' - x)\left((x_i'-x)^2 + (y_i'-y-j)^2\right)^{-1}$$

(List rest of them)

Looking at these equations I found that it would be necessary to know D in order to solve for one derivative in terms of the other. The (x_i', y_i')'s remain in the equation.

For an unknown D fixed on the grid, we increase the size of the grid <u>around</u> D, thereby increasing $-x$, y, x, and $-y$, to see the change in the shape of the $u_D(j)$ curve as this happens.



as $|x|$, $|y|$ increases steadily $u_D(j)$ decreases at each j.

The rate at which $u_D(j)$ decreases decreases as $|x|, |y|$ increases.

" " " " " " depends on j,

i.e. The $u_D(j)$ curve gets flatter as $|x|, |y|$ increases

This information may indicate something about u_x, u_y, u_{-x}, u_{-y}

The "pointedness" of the $u_D(j)$ curve can tell us something about what is happening in the normal direction.

A complicated analysis may arrive at a solution, but this is unlikely because D is unknown, and where D is located affects the shape of the $u_D(j)$ curve, as well.

Sum of corners indicates mass

Another approach to the mass problem is to, generally, consider the overall "size" of $u_D(j)$. This can be done by looking at u at the corners of the grid. Since local min's occur at $j=0$ $j=n$ $j=2n$ $j=3n$ for each $u_i(j)$ they occur there for $u_D(j)$ as well. It is logical to assume that for each m, no matter where the D of this m is located, and configured, the sum $u_D(0)+u_D(n)+u_D(2n)+u_D(3n)$ will be bounded. Looking at this case for $m=1$ we find that that sum is minimized at the center of the grid and maximized at the closest point to a corner. (Proof) Larger D's follow in this pattern, having a location which maximizes and minimizes the sum. If for each m we find a range for this sum, then given the sum alone we can say what are the possible m's that caused it. Continuing in this vein, we can look at the $u_D(j)$ curve, specifically the relationships of u on the sides of the grid to further narrow the range of m.

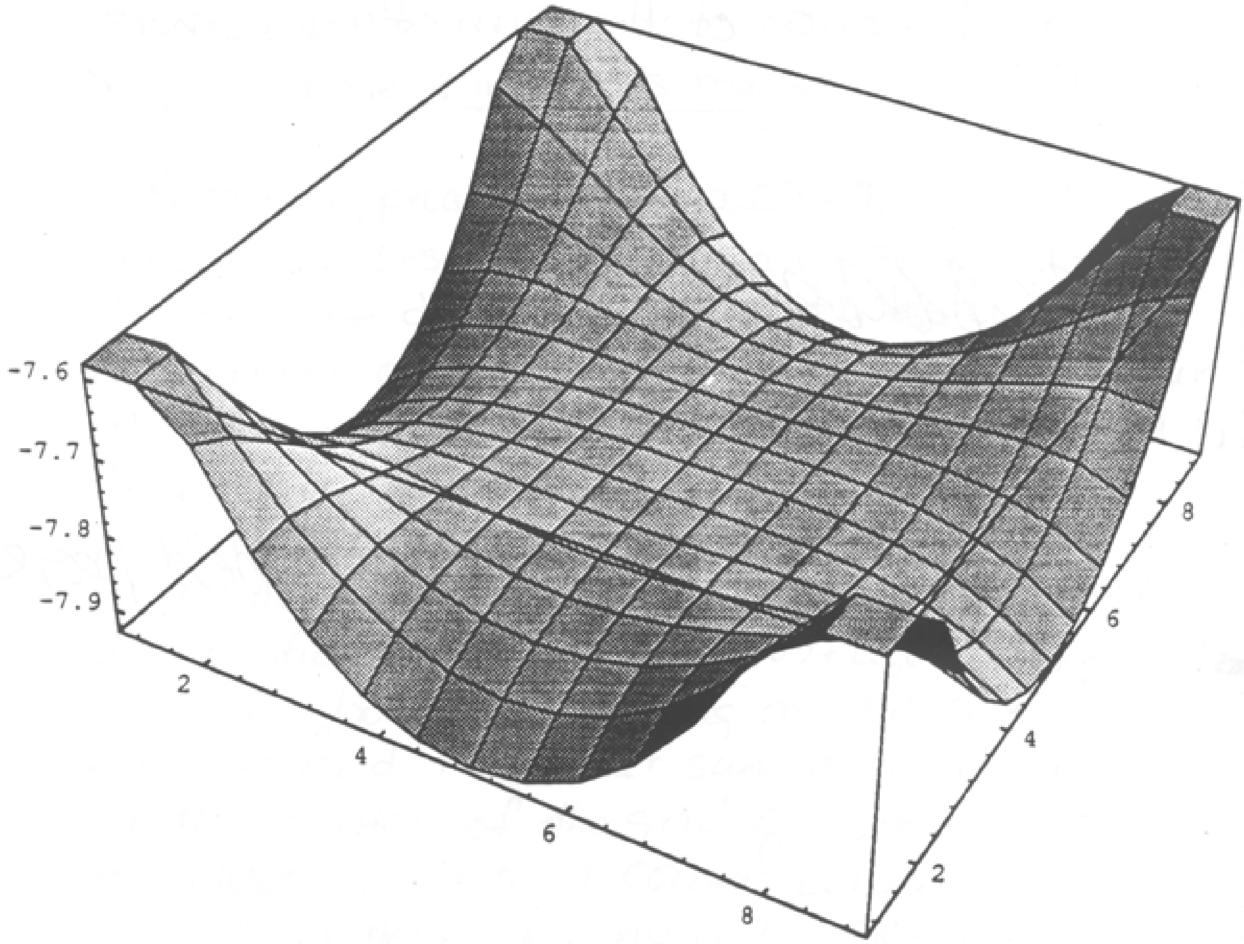
Finding the range of the sum of the corners, brings up an interesting "combinatorial optimization" problem. For a given m, where do you located D so that the sum at the corners is maximized/minimized?

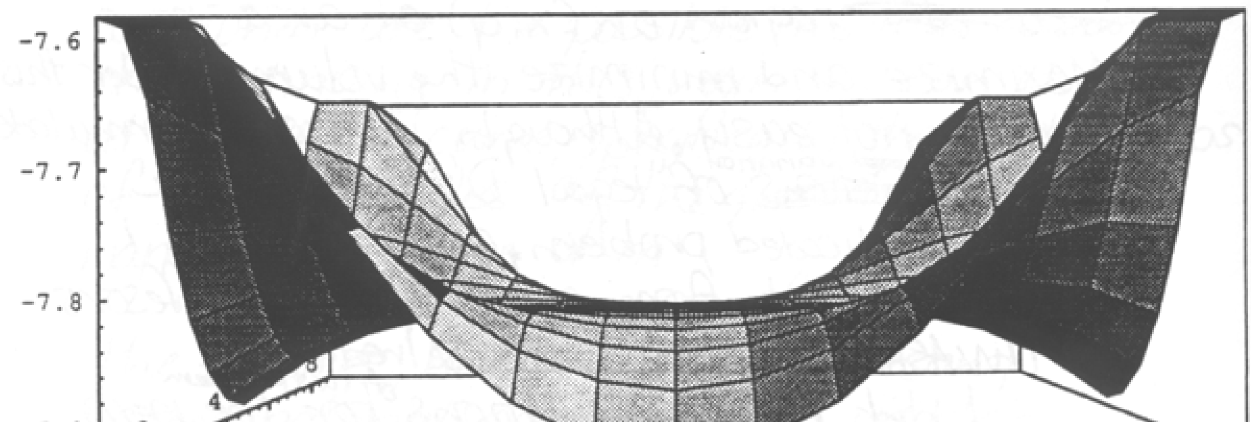
The function of the sum of the corners is:

$\sim$

See illustration next page

Allowed a contiguous, rectangular star-shaped ~~area of~~ region on (x, y) of area = m
Maximize and minimize the volume under this graph. This is not easy, although its easily formulated the ~~restrictions~~ "variations" of legal D's" makes it a very complicated problem, and as far as I could find out from combinatorics professors at UWASH has not been solved yet.
I did hand calculations for the n≤10 case of this problem. The values of $\sim$ ~~at~~ due to ~~all legal~~ D's of m=1 on Ω are located at the corresponding point of Ω.


-7.6
-7.7
-7.8
-7.9
2
4
6
8
2
4
6
8


-7.6
-7.7
-7.8
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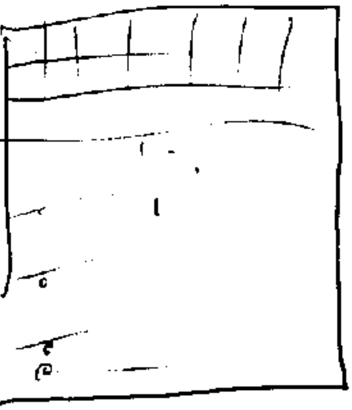


values
1 =
2 =
3 =

To arrived at this table of values.

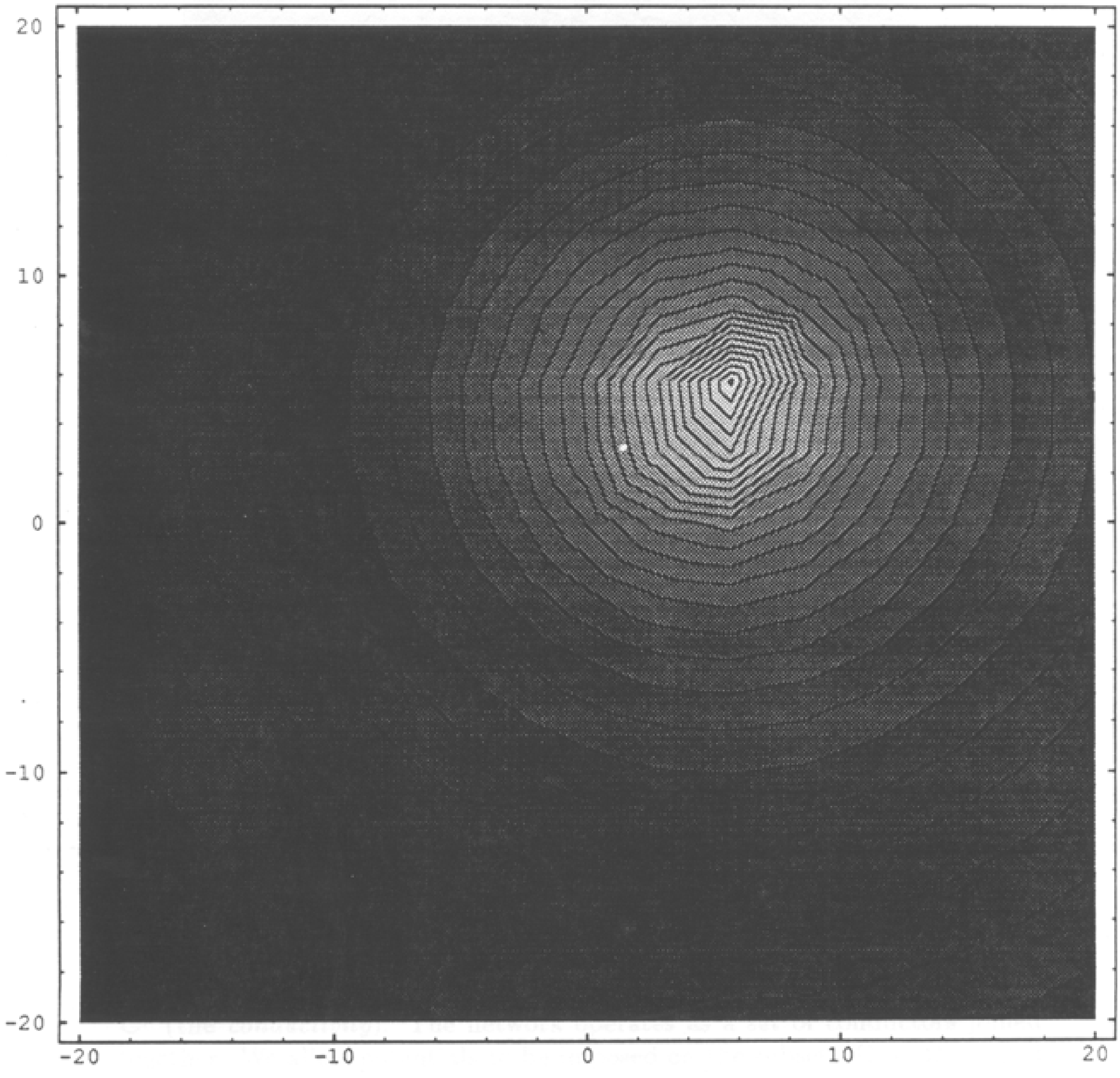
fter m I cannot be sure that
have found the true maximum of the function,
xcept if I were to do ~~tedious calculations of the many~~
ermutations of D of that size.

M bars illustration

The number of "m bars" that any
line ~ = c will pass through is bounded.
Thus the error in finding m from this
method lies, in my estimation from the
(0x10) case, safely within 10%. This does not tell
s much about m, but it tells us something.
his error is ~~the~~ remains the same as n increases
because

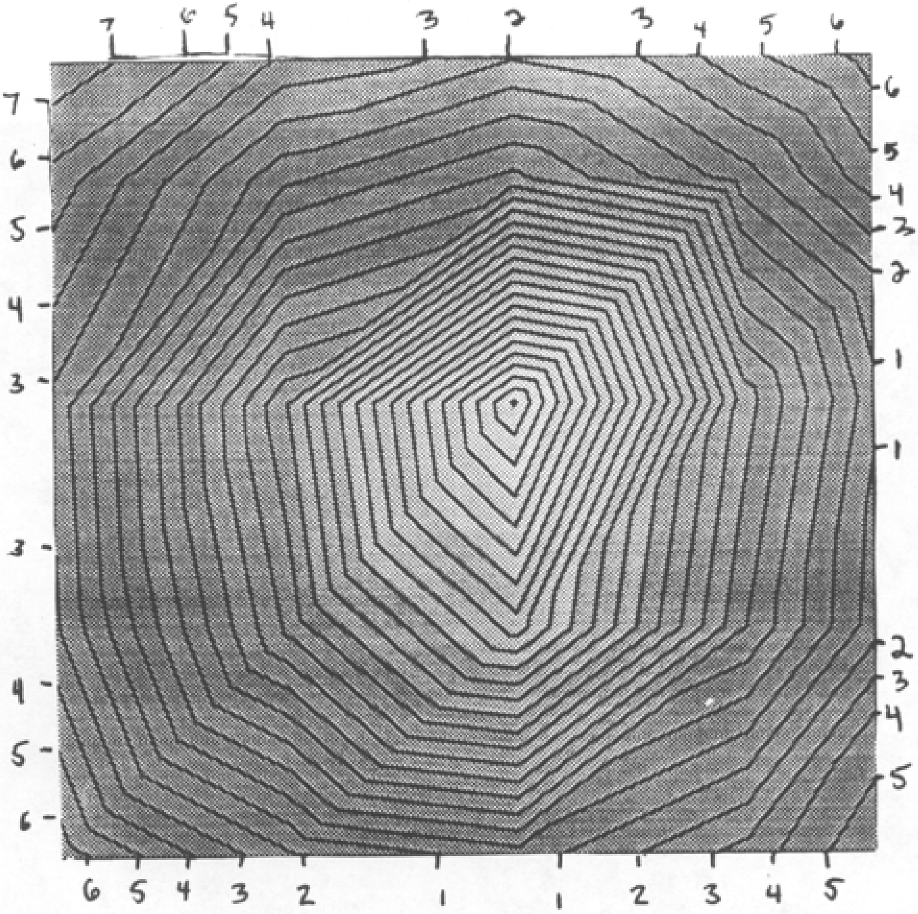
~~if $n_1 = 10$ and $n_2 = 100$~~
then
$u_{P(\ldots)}(i)$ with $n=10$ $= du_{(\ldots)}(j)$ with $n=n_2=n_1$

Finally, I have ~~one~~ an idea which I believe may indicate a solution to the inverse problem, although I have not investigated it fully yet.
Measuring u at discrete ~~on~~ the boundary of the grid and joining the plot of those points, gives an estimation of u measured continuously. From that information we recieve many sets of equal values of u. The sets each lie on equi-potential surfaces caused by D. ~~If D were a point mass~~
For example if D were a point mass, equal values of u would occur at intersections of the grid with concentric circle around D. The distance between these values would vary like the x and y components of $\frac{1}{r}$. From these sets of values we should be able to reconstruct the spherical shape of the equipotential surfaces, and find D. Similarly, I believe the locations of equal-potentials can allow us to reconstruct D.



Equi-potential surfaces from $D = \{(4,5), (5,5), (5,6)\}$

Magnification:



Given the locations of {1}, {2}, {3}, ...
can we reconstruct the surfaces?