Theory of Equivalent Networks and Some of its Applications

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1 Introduction

We consider the following network Ω. The nodes of Ω are the points in 3D space. The set of nodes is denoted by Ω_n. We define some subset Ω_b of Ω_n as *boundary* nodes, so that cardinality of Ω_b $\mathrm{N}_b = card\Omega_b \geq 2$. All other nodes of Ω_n are called *interior*. The set of interior nodes is denoted by Ω_i. $\mathrm{N}_i = card\Omega_i \geq 0$.

$$\Omega_n = \Omega_b \cup \Omega_i.$$
$$\Omega_b \cap \Omega_i = \emptyset.$$
$$\mathrm{N} = card\Omega_n = \mathrm{N}_b + \mathrm{N}_i.$$

For two different nodes p and q belonging to Ω_n the number $\gamma(pq)$ is called *conductance* of pq. The function γ, called *conductivity*, is a non-negative real function. Two nodes p and q are called *neighbors* if $\gamma(pq) > 0$. The set of all neighbors of node p is called *N(p)*. If p and q are neighbors pq is called an *edge* or a *conductor*.

A *network of conductors* is a network with defined set of boundary nodes and conductivity function. Below we will call a network of conductors just a network.

For a node p belonging to Ω_n *u(p)* denotes the *potential* at p. A function u: $\Omega_n \to \mathrm{R}$ gives the *current* from node p to $q \in N(p)$ by

Ohm's Law

$$I(pq) = \gamma(pq)(u(p) - u(q)).$$

For $q \notin N(p)$

$$I(pq) = I(qp) = 0.$$

For a boundary node q

$$I(q) = \sum_{p \in N(q)} I(qp)$$

is called a *boundary current*.

A function u is called *γ-harmonic function*, if for each interior node p

$$\sum_{q \in N(p)} I(pq) = 0.$$

According to *Kirchoff's Law*, if a function ϕ is defined at the boundary nodes, the network Ω will acquire a unique γ-harmonic function u with $u(p) = \phi(p)$ for all boundary nodes p.

For a network Ω with a conductivity function on it the linear map Λ: $\phi \to I_\phi$ is defined by $\Lambda\phi = I_\phi$, where ϕ is a vector of boundary potentials I_ϕ is the corresponding vector of boundary currents. This map is called Dirichlet-to-Neumann map. ¿From the maximum principle and Kirchoff's and Ohm's Laws it can be proved that any Λ has following properties (See [1])

- it is symmetric
- all non-diagonal entries are non-positive
- total of all entries in each row equals 0.

2 Plane equivalent of a network

Definition 2.1 The *Kirchoff matrix* of the network Ω is a square NxN matrix $K(\Omega) = \{k_{ij}\}$ where

$$k_{ij} = -\gamma(p_i p_j) \qquad \text{if} \qquad i \neq j$$

and

$$k_{ii} = \sum_{q \in N(p_i)} \gamma(p_i q).$$

The matrix $K(\Omega)$ contains all information about conductivity function of network Ω. We denote the set of Kirchoff matrices by $\mathcal{K}$. A matrix $M = \{m_{ij}\}$ belongs to $\mathcal{K}$ if

$$\begin{array}{lll} m_{ij} = m_{ji} \leq 0 & \text{if} & i \neq j, \\ m_{ii} > 0, & & \\ \sum_{j=1}^{N} m_{ij} = 0. & & \end{array}$$

By this definition the set of Dirichlet-to-Neumann maps belongs to $\mathcal{K}$.

If $\bar{\Omega}$ is a network with no interior nodes, by Kirchoff's and Ohm's Laws

$$K(\bar{\Omega})\phi = I_\phi,$$

therefore,

$$K(\bar{\Omega}) = \Lambda(\bar{\Omega}).$$

This proves the next statements.

Statement 2.2 The set of Dirichlet-to-Neumann maps *equals* $\mathcal{K}$.

Statement 2.3 For any $\Lambda \in \mathcal{K}$ there exists unique network Ω with no interior nodes such that Λ represents the Dirichlet-to-Neumann map for Ω.

Statement 2.4 The map from networks with no interior nodes to Dirichlet-to-Neumann maps is $1 \leftrightarrow 1$.

Definition 2.5 A *plane equivalent* of network Ω is a network $\bar{\Omega}$ with no interior nodes which has the same Dirichlet-to-Neumann map as Ω.

Theorem 2.6 Any resistor network has unique plane equivalent.

Definition 2.7 Two networks are *equivalent* if they have the same plane equivalent.

For a network Ω with a conductivity function on it the linear map Φ: $\phi \to u_\phi$ is defined by $\Phi\phi = u_\phi$, where ϕ is a vector of boundary potentials and u_ϕ is the corresponding vector of potentials on nodes of Ω. Φ is represented by an $\mathrm{N}\mathrm{x}\mathrm{N}_b$ matrix.

We number nodes of a network Ω so that boundary nodes go first. That is,

$$p_i \in \Omega_b \Leftrightarrow i \leq \mathrm{N}_b$$

and

$p_i \in \Omega_i \Leftrightarrow i > \mathrm{N}_b$.

We write K(Ω) and Φ(Ω) in block form,

$$\Phi(\Omega) = \begin{array}{|c|}\hline I \\ \hline \Phi_\Omega \\ \hline\end{array} \quad \text{and} \quad K(\Omega) = \begin{array}{|c|c|}\hline K' & M^T \\ \hline M & \Xi \\ \hline\end{array}$$

where I is the identity matrix. Now using Kirchoff's and Ohm's Laws we obtain that

$$\begin{array}{|c|c|}\hline K' & M^T \\ \hline M & \Xi \\ \hline\end{array} \; \begin{array}{|c|}\hline I \\ \hline \Phi_\Omega \\ \hline\end{array} = \begin{array}{|c|}\hline \Lambda(\Omega) \\ \hline \mathit{0} \\ \hline\end{array} \tag{1}$$

where *0* is a zero matrix. Since Ξ is non-singular, we can obtain that

$$\Lambda(\Omega) = K(\bar{\Omega}) = K' - M^T \Xi^{-1} M. \tag{2}$$

This equation expresses the Dirichlet-to-Neumann map of a network in terms of blocks of this network's Kirchoff matrix.

3 Blocks in Dirichlet-to-Neumann maps

We consider a network Ω with Dirichlet-to-Neumann map Λ. We write the Kirchoff matrix of Ω in block form,

$$K(\Omega) = \begin{array}{|c|c|}\hline K' & M^T \\ \hline M & \Xi \\ \hline\end{array}$$

Let A be a n by n square submatrix of K' corresponding to rows $i_1, ..., i_n$ and columns $j_1, ..., j_n$. Let C be the N_ixn matrix formed by choosing columns $j_1, ..., j_n$ of block M. Let B be the nxN_i matrix formed by choosing rows $i_1, ..., i_n$ of block M^T. Λ_A is the submatrix of Λ corresponding to rows $i_1, ..., i_n$ and columns $j_1, ..., j_n$.

By formula (2)

$$\Lambda_A = A - B\Xi^{-1}C.$$

Now we have that

$$\begin{array}{|c|c|} \hline A & B \\ \hline C & \Xi \\ \hline \end{array} \;\; \begin{array}{|c|c|} \hline I & 0 \\ \hline -\Xi^{-1}C & I \\ \hline \end{array} = \begin{array}{|c|c|} \hline \Lambda_A & B \\ \hline 0 & \Xi \\ \hline \end{array}$$

If

$$W = \begin{array}{|c|c|} \hline A & B \\ \hline C & \Xi \\ \hline \end{array}$$

we obtain that

$$|W| = |\Lambda_A||\Xi|. \tag{3}$$

Taking A to be an entry of K' we can find any entry of Λ in terms of two determinants of blocks of $K(\Omega)$.

$$\lambda_{ij} = \left| \begin{array}{|c|c|} \hline k'_{ij} & m_i^T \\ \hline m_j & \Xi \\ \hline \end{array} \right| / |\Xi|.$$

Since Ξ is positive definite we have that

$$sign|\Lambda_A| = sign|W|.$$

4 Theorem about elimination of a subnetwork

Definition 4.1 A network $\hat{\Omega} = \{\hat{\Omega}_n, \hat{\Omega}_b, \hat{\gamma}\}$ is a *subnetwork* of $\Omega = \{\Omega_n, \Omega_b, \gamma\}$ if

$$\hat{\Omega}_i \subset \Omega_i.$$

$\hat{\Omega}_b = (\cup_{p \in \hat{\Omega}_i} N(p)) \backslash \hat{\Omega}_i.$
$p, q \in \hat{\Omega}_n \Rightarrow \hat{\gamma}(pq) = \gamma(pq).$

Any subnetwork is defined by its set of interior nodes. The Kirchoff matrix $\hat{K}$ of $\hat{\Omega}$ is a principal submatrix of K.

Definition 4.2 *Elimination* of a subnetwork $\hat{\Omega}$ from a network Ω is a transformation of Ω to Ψ denoted by $\Omega \ominus \hat{\Omega}$ if

$$\begin{array}{lll} \Psi_b = \Omega_b & & \\ \Psi_i = \Omega_i \backslash \hat{\Omega}_i. & & \\ \gamma_\Psi(pq) = \gamma_\Omega(pq) & \text{if} & p \notin \hat{\Omega}_n. \\ \gamma_\Psi(pq) = \gamma_{\hat{\Omega}}(pq) & \text{if} & p, q \in \hat{\Omega}_b. \end{array}$$

where $\gamma_{\hat{\Omega}}$ is the conductivity function on the plane equivalent of $\hat{\Omega}$. In other words to eliminate some subnetwork from a network is to replace this subnetwork with its plane equivalent.

We number the nodes so that the boundary nodes of Ω go first and interior nodes of $\hat{\Omega}$ go last. That is,

$$p_i \in \Omega_b \Leftrightarrow i \leq card\Omega_b$$

and

$$p_i \in \hat{\Omega}_i \Leftrightarrow i > card\Omega - card\hat{\Omega}_i$$

Now we have that

$$K(\Omega) = \begin{array}{|c|c|c|} \hline K' & P^T & U^T \\ \hline P & S & R^T \\ \hline U & R & C \\ \hline \end{array}$$

Taking into account that C is non-singular and using Definition 4.2 and formula (2) we can obtain that if $\Psi = \Omega \ominus \hat{\Omega}$ then

$$K(\Psi) = \begin{array}{|c|c|}\hline K' - U^T C^{-1} U & P^T - U^T C^{-1} R \\ \hline P - R^T C^{-1} U & S - R^T C^{-1} R \\ \hline \end{array} = \Lambda(\{\Psi_n, \Psi_n, \gamma_\Psi\})$$

We define linear maps $\Phi(\Omega)$ and $\Phi(\Psi)$ as it was done in Section 2. Solving equation (1) we obtain that

$$\Phi_\Psi = -(S - R^T C^{-1} R)^{-1}(P - R^T C^{-1} U)$$

and

$$\Phi(\Omega) = \begin{array}{|c|}\hline I \\ \hline \Phi_\Psi \\ \hline -C^{-1}(T + R\Phi_\Psi) \\ \hline \end{array} = \begin{array}{|c|}\hline \Phi(\Psi) \\ \hline -C^{-1}(T + R\Phi_\Psi) \\ \hline \end{array}$$

The non-singularity of $S - R^T C^{-1} R$ follows from (3). Now it is easy to show that

$$\Lambda(\Psi) = \Lambda(\Omega).$$

We proved the following theorem.

Theorem 4.4 Suppose $\Psi = \Omega \ominus \hat{\Omega}$ where $\hat{\Omega}$ is a subnetwork of Ω. Then Ψ and Ω are equivalent. That is $\Lambda(\Psi) = \Lambda(\Omega)$. The matrix $\Phi(\Psi)$ is an upper part of $\Phi(\Omega)$. Therefore,

$$I_\Psi(pq) = I_\Omega(pq) \qquad \text{if} \qquad p, q \notin \Psi_N.$$

In other words, elimination of a subnetwork does not affect potentials and currents outside of it.

5 Recoverable networks

We will say that we know the *shape* of a network if we know all neighbors for each node of this network.

Definition 5.1 A network is *recoverable* if there does not exist a network with the same shape, same plane equivalent (or Dirichlet-to-Neumann map), and different conductivity function.

A network is *non-recoverable* if it is not recoverable.

By Definition 5.1 two recoverable networks with the same shape have different plane equivalents.

Statement 5.2 All subnetworks of a recoverable network are recoverable.

Elimination of node p is an elimination of a subnetwork $\hat{\Omega}$ such that $\hat{\Omega}_i = p$.

Elimination of a subnetwork $\hat{\Omega}$ from Ω results in the same network as elimination of all interior nodes of $\hat{\Omega}$ in any order. Elimination of some set S of interior nodes of Ω results in the same network as elimination of the subnetwork $\hat{\Omega}$ if $\hat{\Omega}_i = S$.

By Theorem 4.4 elimination of an interior node p of a network Ω does not affect plane equivalent of any subnetwork $\hat{\Omega}$ of Ω if $p \in \hat{\Omega}_i$.

6 Chain elimination

We consider a network $\Omega = \{\Omega_n, \Omega_b, \gamma\}$ with $\Omega_b \neq \Omega_n$. We will eliminate interior nodes of Ω one by one. The network obtained after j^{th} elimination is called Ω^j. Let

$$E^j = \Omega_n \backslash \Omega_n^j.$$

Let $\hat{\Omega}^j$ be the subnetwork of Ω with $\hat{\Omega}_i^j = E^j$. Then

$$\Omega^j = \Omega \ominus \hat{\Omega}^j.$$

By Theorem 4.4

$$\Lambda(\Omega) = \Lambda(\Omega^j).$$

Let $\Psi^j = \{\Omega_n, \Psi_b^j, \gamma\}$ where

$$\Psi_b^j = \Omega_n \backslash E^j.$$

$\Psi^{\mathrm{N}_i} = \Omega$. By formula (2) we obtain that

$$K(\Omega \ominus \hat{\Omega}^j) = K(\Omega^j) = \Lambda(\Psi^j).$$

$$K(\Omega) = \Lambda(\Psi^0).$$

7 Specific comments

Formula (2) gives a direct algorithm for finding the Dirichlet-to-Neumann map for any network with known conductivity function. For networks with two boundary nodes this formula represents the generalized parallel and series laws of electrical circuit theory.