Determining Conductances in a ‘Sprinkler’ Network

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1 Introduction

Suppose we need a water supply system in which it is necessary to have various amounts of water exiting. This can be modelled as a network of conductors. The forward problem finds the outflowing currents, given the diameters [1] (conductances) of each of the pipes. These are computed using the pressures (potentials) at each of the nodes and Ohm’s Law. The inverse problem is to find the conductances, given the currents together with $(n-1)^2$ additional parameters. We consider a variation of conductor networks in the plane, as in Curtis and Morrow [2]; however, only square networks will be considered here. We define a network Ω_n that has n^2 nodes and $2n(n-1)$ interior edges; in addition, each node will have a boundary edge. These boundary nodes and edges can be thought of three dimensionally, as sprinkler heads in an actual sprinkler system. There will also be an incoming current, denoted with a dotted line, at one of the interior nodes. Figure 1 shows the $n = 3$ case, Ω_3.

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[1] As a convention, we will use the electricity analog quantities, in parentheses here, in the rest of the paper.

[2] “Determining the Resistors in a Network”, 1990

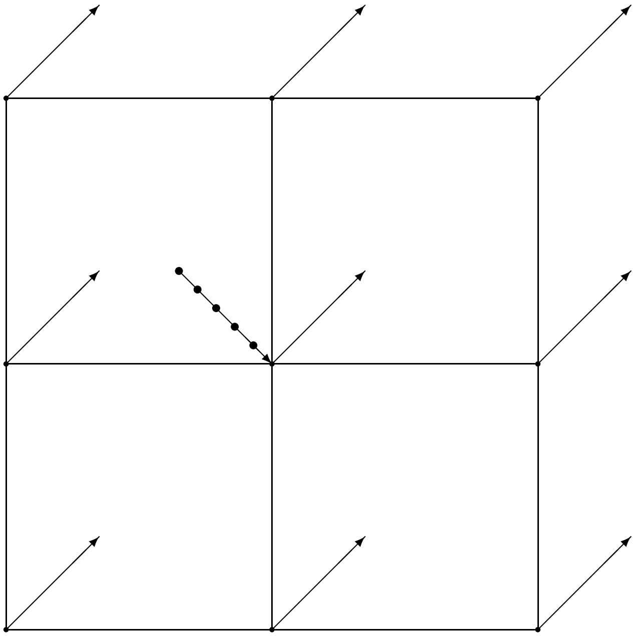

Figure 1

Let E denote the set of edges and let (Ω, γ) be a network of pipes where $\gamma : E \rightarrow R^+$, and $\delta : E \rightarrow R^+$, where R^+ denotes the positve real numbers. For each horizontal edge ϵ, $\gamma(\epsilon)$ is the conductance of ϵ, and for each vertical edge ϵ, $\gamma(\epsilon)$ is the conductance of ϵ.

2 Forward Problem

In order to begin, we need a numbering scheme. For any network, we have n^2 interior nodes denoted N_i. Starting in the upper left corner, we begin with node N_1 and continue across the top to node N_n. Node N_{n+1} starts the second row and so on, until finally, we reach node N_{n^2} in the lower right corner. Each of these nodes is connected to a boundary conductor of conductance 1 and we denote B_i to be the boundary node at the head of the ith boundary conductor. We let γ_i be the conductance of the interior horizontal edges and δ_j be the conductance of the interior vertical edges. Like the nodes, they are numbered left to right.

We set the conductance at each boundary edge to 1 and the potential at each boundary node to 0; consequently, the interior potential and the boundary current are equal, by Ohm's Law. Our numbering scheme and given conditions are illustrated in Figure 2. The potentials at each boundary node are equal to 0. We also show a unit incoming current at node 5, denoted with parentheses.

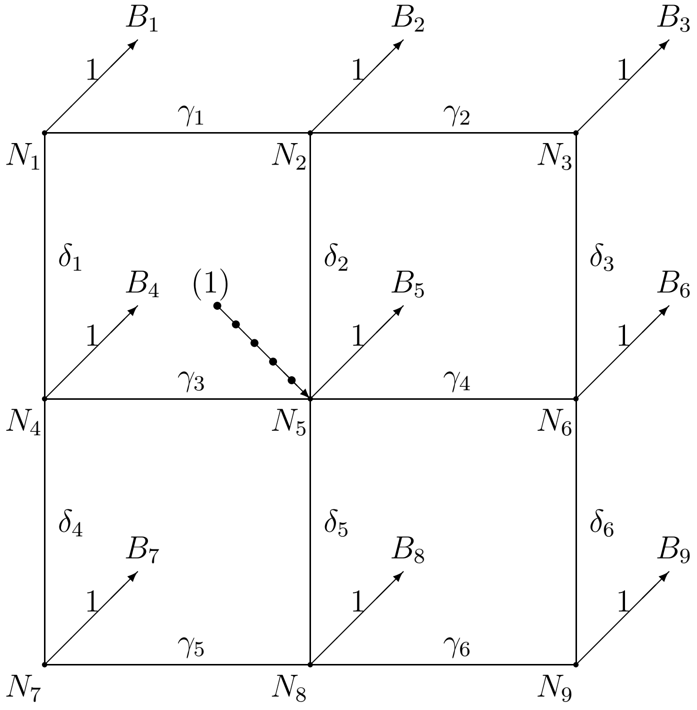
B_1
B_2
B_3
1
1
1
γ_1
γ_2
N_1
N_2
N_3
δ_1
δ_2
δ_3
B_4
(1)
B_5
B_6
1
1
1
γ_3
γ_4
N_4
N_5
N_6
δ_4
δ_5
δ_6
B_7
B_8
B_9
1
1
1
γ_5
γ_6
N_7
N_8
N_9

Figure 2

There is a system of n^2 equations obtained by applying Kirchhoff's Law to each interior node. In matrix form this can be expressed as $Ku = b$, where we define a matrix K, which organizes the given conductances from the system of equations. The diagonal entry is the sum of all the neighboring conductances, the row number of an entry corresponds to the node number and the column numbers represent relationships between nodes. For example, $k_{1,7} = 0$ means that with respect to node one, node seven is not connected and

$k_{3,2} = -\gamma_2$ means that node three is connected to node two with a conductor of conductance γ_2, where the minus sign is a consequence of Ohm's Law.

Using the equation $Ku = b$ we can solve for the potentials at each interior node, given the conductance of each interior edge. We define u to be the vector of interior potentials and b to be the vector of boundary potentials with the incoming current. Assume a unit incoming current at node 5. Once we solve for the potentials at the interior nodes, we have the outflowing currents since the boundary conductances are 1 and the boundary potentials are 0. In the case of Ω_3, Kirchhoff's Law at node 5 gives

$$1(u_5 - 0) + \delta_2(u_5 - u_2) + \gamma_3(u_5 - u_4) + \gamma_4(u_5 - u_6) + \delta_5(u_5 - u_8) = 1$$

The fifth line of the matrix K is:

$$0 \quad -\delta_2 \quad 0 \quad -\gamma_3 \quad 1 + \delta_2 + \gamma_3 + \gamma_4 + \delta_5 \quad -\gamma_4 \quad 0 \quad -\delta_5 \quad 0$$

Continuing this way, the matrix is:

$$\begin{bmatrix}
1+\gamma_1+\delta_1 & -\gamma_1 & 0 & -\delta_1 & 0 & 0 & 0 & 0 & 0 \\
-\gamma_1 & 1+\gamma_2+\delta_2 & -\gamma_2 & 0 & -\delta_2 & 0 & 0 & 0 & 0 \\
0 & -\gamma_2 & 1+\gamma_2+\delta_3 & 0 & 0 & -\delta_3 & 0 & 0 & 0 \\
-\delta_1 & 0 & 0 & \begin{matrix}1+\delta_1+\gamma_3\\+\delta_4\end{matrix} & -\gamma_3 & 0 & -\delta_4 & 0 & 0 \\
0 & -\delta_2 & 0 & -\gamma_3 & \begin{matrix}1+\delta_2+\gamma_3\\+\gamma_4+\delta_5\end{matrix} & -\gamma_4 & 0 & -\delta_5 & 0 \\
0 & 0 & -\delta_3 & 0 & -\gamma_4 & \begin{matrix}1+\delta_3+\gamma_4\\+\delta_6\end{matrix} & 0 & 0 & -\delta_6 \\
0 & 0 & 0 & -\delta_4 & 0 & 0 & 1+\delta_4+\gamma_5 & -\gamma_5 & 0 \\
0 & 0 & 0 & 0 & -\delta_5 & 0 & -\gamma_5 & \begin{matrix}1+\delta_5+\gamma_5\\+\gamma_6\end{matrix} & -\gamma_6 \\
0 & 0 & 0 & 0 & 0 & -\delta_6 & 0 & -\gamma_6 & 1+\delta_6+\gamma_6
\end{bmatrix}$$

Notice that the matrix K is symmetric and banded. Using K, we can solve the forward problem for any set of conductances by solving for u in the equation $Ku = b$. This can be solved using a computer program [3].

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[3] See attached Fortran program, AMATRIX

3 Inverse Problem

In the forward problem, we are given conductances and define potentials on the boundary. From that, we determine the interior potentials and hence, the outflowing currents. For the inverse problem, we begin with known current flows and boundary potentials, and wish to recover the conductances which satisfy these conditions. In the 2×2 case, (see Figure 3) it appears that we have four outflowing currents and four conductances, thus completely determining the solution. However, there are really only three equations that are given since the fourth current can be determined by the other three. Therefore, even in the simplest case we only have three equations and four unknowns, giving us one degree of freedom.

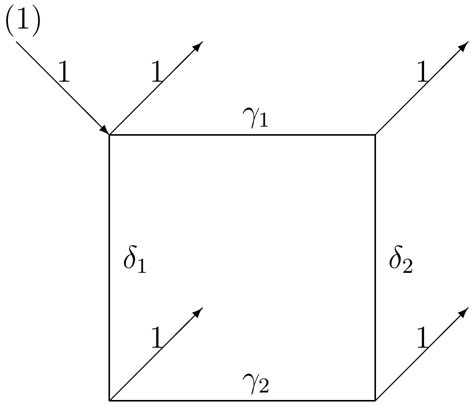

Figure 3

For any Ω_n, we will always have $n^2 - 1$ independent equations from Kirchhoff's Law at the interior nodes, rather than n^2 giving us one more degree of freedom than anticipated. The number of degrees of freedom in any system is $(n-1)^2$. We define a matrix M, which organizes the potentials in order to compute the conductances. We would like to solve the equation $Mx = b$ where x is the vector of unknown conductances and b is the same as in the forward problem. For this type of network, we need to recover more conductances than there are boundary currents; therefore M will not be square.

For instance, in the 3×3 case, we are given nine currents in order to recover twelve conductances, giving a 9×12 matrix M.

4 Method for Solving the Inverse Problem

The basic idea behind the inverse algorithm is to first determine Kirchhoff's equations for each interior node in the network. Then we use these equations to create the matrix M, choose $(n-1)^2$ valid parameters and reduce the matrix for these parameters. The values for each of the γ_i and δ_j's can be easily read off the reduced matrix. We illustrate this with the following cases.

4.1 The 2×2 Case

In order to solve the inverse problem, we again begin with Kirchhoff's Law and the resulting equations for each node. For example, in the 2×2 case, (see Figure 3) we have the following equations:

$$\begin{aligned}
u_1 + \gamma_1(u_1 - u_2) + \delta_1(u_1 - u_3) &= 1 \qquad (1)\\
u_2 + \gamma_1(u_2 - u_1) + \delta_2(u_2 - u_4) &= 0 \qquad (2)\\
u_3 + \delta_1(u_3 - u_1) + \gamma_2(u_3 - u_4) &= 0 \qquad (3)\\
u_4 + \delta_2(u_4 - u_2) + \gamma_2(u_4 - u_3) &= 0 \qquad (4)
\end{aligned}$$

Writing these in matrix form in terms of γ's and δ's, we have:

$$\begin{bmatrix} u_1 - u_2 & u_1 - u_3 & 0 & 0 \\ u_2 - u_1 & 0 & u_2 - u_4 & 0 \\ 0 & u_3 - u_1 & 0 & u_3 - u_4 \\ 0 & 0 & u_4 - u_2 & u_4 - u_3 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \delta_1 \\ \delta_2 \\ \gamma_2 \end{bmatrix} = \begin{bmatrix} 1 - u_1 \\ -u_2 \\ -u_3 \\ -u_4 \end{bmatrix}$$

Rewriting this in augmented form and reducing by Gauss-Jordan elimination, yields:

$$\begin{bmatrix} u_1 - u_2 & 0 & 0 & u_3 - u_4 & \vdots & u_2 + u_4 \\ 0 & u_3 - u_1 & 0 & u_3 - u_4 & \vdots & -u_3 \\ 0 & 0 & u_2 - u_4 & u_3 - u_4 & \vdots & u_4 \\ 0 & 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

Notice that one row is reduced to all zeros, leaving one of the conductors as a parameter. Given potentials at each node, the equations determined by the matrix to recover the γ's and δ's are

$$\gamma_1 = \left(\frac{u_1 + u_4}{u_1 - u_2}\right) - \gamma_2 \left(\frac{u_3 - u_4}{u_1 - u_2}\right) \tag{5}$$

$$\delta_1 = \left(\frac{u_4}{u_2 - u_4}\right) - \gamma_2 \left(\frac{u_3 - u_4}{u_2 - u_4}\right) \tag{6}$$

$$\delta_2 = \left(\frac{-u_3}{u_3 - u_1}\right) - \gamma_2 \left(\frac{u_3 - u_4}{u_3 - u_1}\right) \tag{7}$$

where γ_2 is the parameter.

4.2 The 3×3 Case

A similar solution will be obtained for the 3×3 case. For this case, we begin with nine equations and need to recover twelve conductances. Again, there are only eight independent equations since the ninth potential can be obtained from the others. This leaves four parameters in the inverse solution. This solution is found by putting the Kirchhoff equations in an augmented matrix and solving the system. We choose values for the parameters, $\delta_1, \delta_3, \delta_4$, and δ_6. Then, the remaining conductances are defined by the following equations:

$$\gamma_1 = \left(\frac{-u_1}{u_1 - u_2}\right) - \delta_1 \left(\frac{u_1 - u_4}{u_1 - u_2}\right) \tag{8}$$

$$\gamma_2 = \left(\frac{u_3}{u_2 - u_3}\right) + \delta_3 \left(\frac{u_3 - u_6}{u_2 - u_3}\right) \tag{9}$$

$$\gamma_3 = \left(\frac{-u_4}{u_4 - u_5}\right) - \delta_1 \left(\frac{u_4 - u_1}{u_4 - u_5}\right) - \delta_4 \left(\frac{u_4 - u_7}{u_4 - u_5}\right) \tag{10}$$

$$\gamma_4 = \left(\frac{u_6}{u_5 - u_6}\right) - \delta_3 \left(\frac{u_3 - u_6}{u_5 - u_6}\right) - \delta_6 \left(\frac{u_9 - u_6}{u_5 - u_6}\right) \tag{11}$$

$$\gamma_5 = \left(\frac{-u_7}{u_7 - u_8}\right) + \delta_4 \left(\frac{u_4 - u_7}{u_7 - u_8}\right) \tag{12}$$

$$\gamma_6 = \left(\frac{u_9}{u_8 - u_9}\right) + \delta_6 \left(\frac{u_6 - u_9}{u_8 - u_9}\right) \tag{13}$$

$$\delta_2 = \left(\frac{-u_1 - u_2 - u_3}{u_2 - u_5}\right) - \delta_1 \left(\frac{u_1 - u_4}{u_2 - u_5}\right) - \delta_3 \left(\frac{u_3 - u_6}{u_2 - u_5}\right) \tag{14}$$

$$\delta_5 = \left(\frac{u_7 + u_8 + u_9}{u_5 - u_8}\right) - \delta_4 \left(\frac{u_4 - u_7}{u_5 - u_8}\right) - \delta_6 \left(\frac{u_6 - u_9}{u_5 - u_8}\right) \tag{15}$$

There are a number of options in choosing the parameters. It is advantageous to choose parameters with some symmetry within the network, as in the above case. However, there are some restrictions based on Kirchhoff's Law. For instance, we can't choose the center four conductors (shown in Figure 4 as double lines) to be the parameters because once three have been chosen, the fourth is automatically determined. Also, it can be seen on the diagram that none of the other conductances can be found by Kirchhoff's Law. In order to determine a conductance, all of the neighboring conductances must be previously determined. For instance, if we want to find γ_1, we either need to know δ_1, or we need to know γ_2 and δ_2 simultaneously, which we don't have in either case.

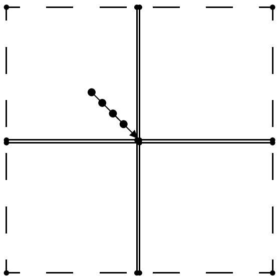

Figure 4

5 Characterization of the Inverse Solution

Since we have a solution involving one or more parameters, it appears that we have an infinite set of solutions; however, we must be cautious because each γ_i and δ_j must be positive, thus greatly reducing our set of possible solutions. Another factor that must be considered is the direction of current flow within the network. Depending upon which case we have we can get completely different sets of solutions. For any of the inverse problems, there is no one unique solution, but rather a unique solution for any given set of parameters.

5.1 The 2 × 2 Case

The 2 × 2 case is the simplest network to consider. Here, there are three cases.
Case I: (see Figure 5)

$$u_1 > u_2$$

$$u_1 > u_3$$

$$u_2 > u_4$$

$$u_3 > u_4$$

Case II: (see Figure 6)

$$u_1 > u_2$$

$$u_1 > u_3$$

$$u_3 > u_4$$

$$u_4 > u_2$$

and Case III: (see Figure 7)

$$u_1 > u_2$$

$$u_1 > u_3$$

$$u_2 > u_4$$

$$u_4 > u_3$$

The flow patterns are indicated by the arrows.

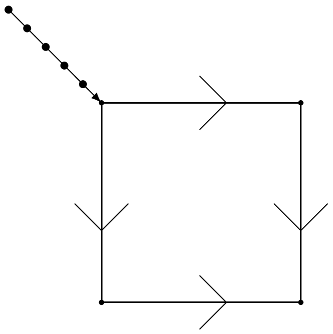

Figure 5

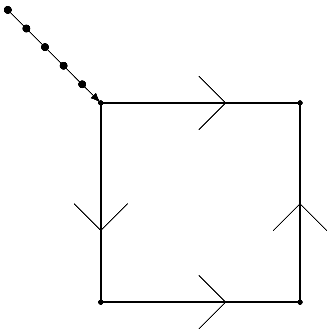

Figure 6

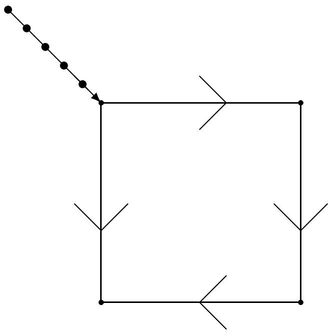

Figure 7

Notice that the current flow is away from the node in which there is an source current. This will always occur for any size network. To explain this, suppose that there was one current flow into node N_1 and one flowing out. For this to happen, the currents must flow in a clockwise manner. The inequality which describes this is: $u_1 > u_2 > u_4 > u_3 > u_1$; a contradiction since we can't have $u_1 > u_1$. Therefore, the current flow at a node with an incoming current will always be away from the node, making it the node with the highest potential.

When we analyze the equations (1) through (4), in $(\gamma_1, \gamma_2, \delta_1, \delta_2)$–space, we find that for each case there is a unique line-segment in which the possible sets of γ's and δ's lie in order to satisfy the given currents.

For instance, in Case I we must determine a set of inequalities to limit the parameter values. Beginning with equations (5) through (7) and knowing that γ_i and δ_j, for all i and j, must always be positive, we find that

$$\left(\frac{u_2 + u_4}{u_3 - u_4}\right) > \gamma_2 \tag{16}$$

$$\left(\frac{-u_3}{u_3 - u_4}\right) < \gamma_2 \tag{17}$$

$$\left(\frac{u_4}{u_3 - u_4}\right) > \gamma_2 \tag{18}$$

This situation reduces to the following simple inequality:

$$0 < \quad \gamma_2 < \quad \left(\frac{u_4}{u_3 - u_4}\right) \tag{19}$$

It would be more convienient to have the equations in terms of an arbitrary parameter t, so now we reparametrize. Using (19), we can find the endpoints of the line segment solution set in $(\gamma_1, \gamma_2, \delta_1, \delta_2)$– space by first letting $\gamma_2 = 0$ and then letting $\gamma_2 = \frac{u_4}{u_3 - u_4}$. This gives us the points

$$\left(\frac{u_2 + u_4}{u_1 - u_2}\ ,\ \frac{-u_3}{u_3 - u_1}\ ,\ \frac{u_4}{u_2 - u_4}\ ,\ 0\right) \text{ and } \left(\frac{u_1}{u_1 - u_2}\ ,\ \frac{-u_3 - u_4}{u_3 - u_1}\ ,\ 0\ ,\ \frac{u_4}{u_3 - u_4}\right).$$

To reparametrize a line, we need a point and a direction vector. Finding the midpoint of these two points and subtracting the first point from the midpoint gives us this vector:

$$\left(\frac{-u_4}{2(u_1 - u_2)}\ ,\ \frac{-u_4}{2(u_3 - u_1)}\ ,\ \frac{-u_4}{2(u_2 - u_4)}\ ,\ \frac{u_4}{2(u_3 - u_4)}\right)$$

Since we have a point and a vector, the new parametrization can easily be determined. It is:

$$\begin{aligned}
\gamma_1 &= \frac{2u_2 + u_4}{2(u_1 - u_2)} - t\left(\frac{-u_4}{2(u_1 - u_2)}\right) && (20)\\
\gamma_2 &= \frac{u_4}{2(u_3 - u_4)} + t\left(\frac{u_4}{2(u_3 - u_4)}\right) && (21)\\
\delta_1 &= \frac{-2u_3 - u_4}{2(u_3 - u_1)} - t\left(\frac{u_4}{2(u_3 - u_1)}\right) && (22)\\
\delta_2 &= \frac{u_4}{2(u_2 - u_4)} - t\left(\frac{u_4}{2(u_2 - u_4)}\right) && (23)
\end{aligned}$$

where

$$\frac{2u_3 + u_4}{-u_4} < t < 1.$$

Similar equations and inequalities occur for Case II and Case III and are determined in the same manner.

Now that we have the solution set, we can do a variety of different things. For instance, if we are given a set of conductances and we wish to know if they fall within the solution set, we simply have to check if the given conductances satisfy the equations for the parametrized line. We can also minimize the diameters of the pipes by using basic calculus. Say, for instance, we want to minimize $d = \gamma_1^2 + \gamma_2^2 + \delta_1^2 + \delta_2^2$ we simply substitute the equations for each γ and δ in terms of t, take the derivitive, set it equal to 0 and solve for t.

5.2 The 3 × 3 Case

We use the same method to characterize the 3 × 3 case. Suppose we have a unit current entering at node N_5. Then there are a great number of possible current flows that we could consider. However, we know that the current flow must be away from the source node, which leaves a choice of Case I, II, or III from the 2×2 system in each of the four corners. Therefore, there are eighty-one different current flow patterns to choose from, although many are the same because of symmetry and some are not possible. We will discuss a typical case where the following inequalites hold: (see Figure 8)

$$\begin{array}{lclclcl} u_1 & < & u_2 & \qquad & u_3 & < & u_2 \\ u_1 & < & u_4 & & u_2 & < & u_5 \\ u_3 & < & u_6 & & u_4 & < & u_5 \\ u_6 & < & u_5 & & u_7 & < & u_4 \\ u_8 & < & u_5 & & u_9 & < & u_6 \\ u_7 & < & u_8 & & u_9 & < & u_8 \end{array}$$

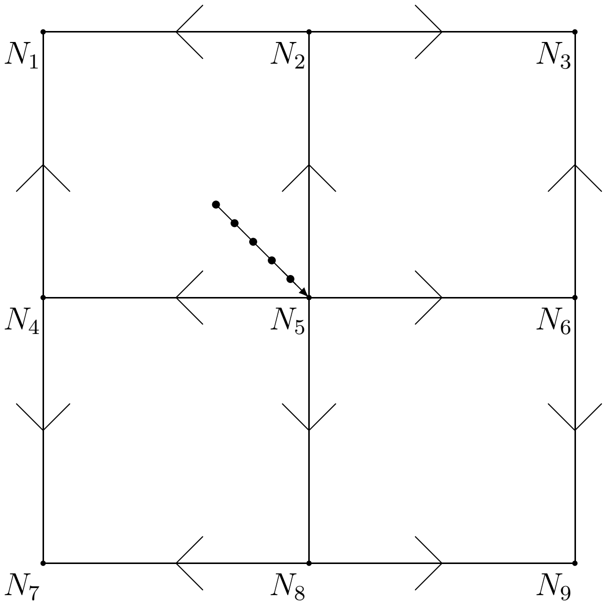

Figure 8

Referring back to equations (8) through (15), the previous inequalities, and the conditions that $\gamma_i > 0$ and $\delta_j > 0$, for all i and j, we can generate the following inequalities:

$$\left(\frac{u_1}{u_4 - u_1}\right) > \delta_1 \tag{24}$$

$$\left(\frac{u_3}{u_6 - u_3}\right) > \delta_3 \tag{25}$$

$$\left(\frac{u_4}{u_1 - u_4}\right) - \delta_4 \left(\frac{u_7 - u_4}{u_1 - u_4}\right) < \delta_1 \tag{26}$$

$$\left(\frac{u_6}{u_3 - u_6}\right) - \delta_6 \left(\frac{u_9 - u_6}{u_3 - u_6}\right) < \delta_3 \tag{27}$$

$$\left(\frac{u_9}{u_6 - u_9}\right) > \delta_6 \tag{28}$$

$$\left(\frac{u_1 + u_2 + u_3}{u_4 - u_1}\right) - \delta_3 \left(\frac{u_6 - u_3}{u_4 - u_1}\right) > \delta_1 \tag{29}$$

$$\left(\frac{u_7+u_8+u_9}{u_4-u_7}\right)-\delta_6\left(\frac{u_6-u_9}{u_4-u_7}\right) > \delta_4 \tag{30}$$
$$\left(\frac{u_7}{u_4-u_7}\right) > \delta_4 \tag{31}$$

These inequalities can be reduced even further. Notice that the left side of (26) is always negative; therefore, this inequality is unnecessary since we must have $\delta_1 > 0$. This is also the case for (27). Next we want to compare (24), (25), and (29). Substituting both 0 and equation (25) into (29) gives us the following:

$$\left(\frac{u_1+u_2+u_3}{u_4-u_1}\right) > \delta_1 \text{ and} \tag{32}$$
$$\left(\frac{u_1+u_2}{u_4-u_1}\right) > \delta_1. \tag{33}$$

All other values of δ_3 will fall in between these two values. However, these are both larger than the value in (24), therefore, we need not be concerned with (29). A similar argument follows for (28), (30), and (31).

The list of inequalities reduces to the following:

$$0 < \delta_1 < \left(\frac{u_1}{u_4-u_1}\right) \tag{34}$$
$$0 < \delta_3 < \left(\frac{u_3}{u_6-u_3}\right) \tag{35}$$
$$0 < \delta_4 < \left(\frac{u_7}{u_4-u_7}\right) \tag{36}$$
$$0 < \delta_6 < \left(\frac{u_9}{u_6-u_9}\right). \tag{37}$$

For the 3×3 case, the solution lies in $(\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5,\gamma_6,\delta_1,\delta_2,\delta_3,\delta_4,\delta_5,\delta_6)$–space in $\mathbf{R}^{12}$. The above inequalities produce a 4-dimensional rectangular solution space in $\mathbf{R}^{12}$; therefore, the possible sets of γ_i and δ_j's lie inside this region.

6 Stability

We are capable of recovering conductances given desired outflowing currents. We are interested in how accurately we can obtain the conductances when we start with a small percentage error in the given currents. We choose arbitrary current flows for a 3 × 3 case that are accurate to six digits. We then change the currents by .001, .0001, and .00001 respectively. This causes the percentchange to fluctuate between each of the cases, therefore we also calculate the conditional number, defined as

$$\left(\frac{\frac{\Delta current}{current}}{\frac{\Delta conductance}{conductance}}\right).$$

This is shown in Table I.

Node number changed	*Current*	*Maximum % error (.001)*	*Maximum% error (.0001)*	*Maximum% error (.00001)*	*Conditional number*
1	.014595	21.25	2.15	.213	.322
2	.082171	1.50	.20	.020	.813
3	.031176	46.07	4.69	.469	.070
4	.105790	13.70	1.37	.137	.069
5	.376551	.65	.07	.006	.409
6	.221985	5.87	.59	.059	.077
7	.018848	82.46	8.35	.836	.064
8	.091263	3.06	.30	.030	.359
9	.057621	18.08	1.86	.186	.096

Table I

Notice that the greatest maximum errors occur at the corner nodes, and the smallest maximum error occurs at the center node. Thus, the further away we get from the center the larger the error. When we make a change of .001 in one of the nodes, the percent errors are much too big. Changes of .0001 are respectable unless the desired current flow is fairly small as in node 7. A change of .00001 gives an acceptable error although the percent changes for this case are between .0005% and .836%, which is a difficult task to accomplish. Therefore, in order for the inverse solutions to be accurate we have a rather small margin of error in our given current flows.

7 Conclusion

The cases we covered in depth are specific, however, these methods can be generalized to other cases and bigger networks. Although as the network size increases, there are exponentially more possible flow patterns. We also looked briefly at cases where the source current occured at various nodes. these cases were more complicated due to the asymmetry of the network. we also considered multiple currents; however, this wasn't pursued because there was little variation from the single source network.