Using Iterative Methods to Solve the Dirichlet and Inverse Problems

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August 16, 1991

Abstract

In this paper I compare different methods to solve the Dirichlet problem. I concluded that the Gauss-Seidel method is the most efficient of these methods. Using this method, I can determine the values of the interior resistors in a 3-dimensional network given measurements at the boundary.

The Dirichlet Problem

The Dirichlet problem involves determining inner values from given boundary conditions. In the continuous case, it is solved using Laplace's differential equation,

$$U_{xx} + U_{yy} = 0$$

However, in this paper I will discuss the discrete Dirichlet problem which is solved using Kirchoff's Law, the discrete formulization of Laplace's equation.

Given an $n \times n$ electrical network with known inner conductances and boundary potentials, the Dirichlet problem involves finding the inner potentials for this network.(see fig. 1)

Let $\gamma(PQ)$ be the conductance between interior nodes P and Q. Thus, from Kirchoff's Law,

$$\sum_{Q neighbor of P} \gamma(PQ)(U(Q) - U(P)) = 0$$

and,

$$[\sum_{Q neighbor of P} \gamma(PQ)]U(P) = \sum_{Q neighbor of P} \gamma(PQ)U(Q).$$

We can then determine the values of the potentials at the interior nodes by solving the system of linear equations,

$$Au = b$$

Here, A is the $n^2 \times n^2$ Kirchoff matrix and u is the vector of interior potentials. Therefore, the solution of this system of linear equations is the solution to the Dirichlet problem.

The Inverse Problem

The inverse problem involves determining boundary values from given inner values. In order to solve this problem, the Dirichlet-to-Neumman mapping is used.

After finding the inner potentials by solving the Dirichlet problem, we can then find Λ which is the Dirichlet-to -Neumann mapping from potentials on the boundary to currents on the boundary. The current into node P due to the potential at boundary node P' is

$$\gamma(PP')(U(P) - U(P')).$$

Since we know the inner and boundary potentials of our network we can determine each element in Λ. An entry in Λ, $\mathrm{I}_{i,j}$, is the current at node i due to a potential of 1 at node j. Thus,

$$\Lambda u = i$$

where i is the column vector containing the values of the currents at the boundary nodes. The inverse problem can then be solved using Λ as discussed in Curtis and Morrow (1).

The Jacobi Iterative Method

In order to decrease the computing time required to solve the Dirichlet problem, I decided to try using iterative methods. The first iterative method I studied was the Jacobi iterative method. Once again we want to obtain the solution of the system of linear equations,

$$Au = b$$

where A is the Kirchoff matrix, u is the vector of inner potentials, and b is determined from the values of the boundary potentials. The Kirchoff matrix ,A , can be expressed as a matrix sum of its diagonal entries and its off diagonal entries. Namely,

$$A = D + N,$$

where D is the matrix with diagonal entries and zeroes elsewhere, and N is the matrix with off diagonal entries and zeroes along the main diagonal. Thus, we can express our linear system as follows:

$$Au = b$$

can be written as,

$$(D + N)u = b.$$

Therefore,

$$Du = b - Nu.$$

Since A has diagonal entries which are nonzero, we can use the following iterative method.

$$a_{i,i}u_i^{(m+1)} = b_i - \sum_{j=1, j\neq i}^{n} a_{i,j}u_j^{(m)}$$

or in matrix form,

$$Du^{(m+1)} = b - Nu^{(m)}$$

where we have an initial guess u_0. According to Smith (6), since A is diagonally dominant this iterative method will converge. This method is the Jacobi iterative method.

The Gauss-Seidel Iterative Method

When using the Jacobi iterative method one must store all of the values in the vector $u^{(m)}$ to compute the values of the vector $u^{(m+1)}$. Therefore, I decided to examine the Gauss-Seidel iterative method. In the Gauss-Seidel method, no previous values need to be stored. The algorithm for the Gauss-Seidel method is the following:

$$u_i^{(m+1)} a_{i,i} = b_{i,i} - \sum_{j=1}^{i-1} a_{i,j} u_j^{(m+1)} - \sum_{j=i+1}^{n} a_{i,j} u_j^{(m)}$$

Again, this procedure is repeated until the values of u converge. By Theorem 2.4 in Johnson and Riess (3), u converges for any initial guess $u^{(0)}$. This method of iteration converges more rapidly then the Jacobi method since values are used as soon as they are calculated. This can be seen in the graphs included in the appendix.

The Gauss-Seidel Iterative Method In 3 Dimensions

After discovering the efficiency of using the Gauss-Seidel method to solve the Dirichlet problem in a 2-dimensional network, I considered applying it to solving the Dirichlet problem in three dimensions. The complications and the great amount of computing time that arise to solve the system of linear equations,

$$Au = b,$$

in 3 dimensions using linear algebra persuaded me to use this method. The Gauss-Seidel method succeeded in significantly decreasing the complexity and the computing time in solving the Dirichlet problem in 3 dimensions.

Calculating Isolated Resistors in a 3-Dimensional Network

Since the dirichlet problem in 3 dimensions can be solved efficiently, I decided to try to find an algorithm similar to Landrum's algorithm in 2 dimensions (5) to determine the value of an isolated resistor in a 3-dimensional network. This algorithm involves isolating a resistor by forcing the boundary currents from one face to flow through an interior resistor.

The first step is to make conditions on the boundary to restrict current from the other faces to flow through the isolated resistor. The value of the isolated resistor is then determined by summing the values of the boundary currents from the single face. The following is an example of isolating a resistor by making a specific set of boundary conditions.

Kirchoff's Law determines some of the interior potentials from the values of the boundary conditions as shown above. This in turn forces the isolated resistor to be equal to the sum of the currents from the top face.

If you consider the shape formed from the interior nodes which have a potential of one, you will notice a pyramidal structure. The same is true for those interior nodes which have a potential of zero. These same shapes will occur when resistors are isolated in other 3-dimensional networks.

¿From the boundary conditions given you can set up the following system of equations.

$$(\sum_{l=1}^{unknown no. of potentials} v_l I_j)_i + (\sum I_k)_i = 0$$

where j corresponds to the position of the boundary node where the potential is not known, i corresponds to the position of the boundary node where there is no current, and k corresponds to the position of the boundary node where the potential is equal to 1.

The I's represent the elements in the Kirchoff matrix, Λ. In the example, there will be 30 equations and 30 unknowns. After solving this system of equations, the potentials at the face from which the current that flows through the isolated resistor originates from are determined. Using the Dirichlet-to -Neumann mapping, Λ, we can find the currents at the boundary of this face since

$$\Lambda u = i$$

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The value of the isolated resistor is determined by summing these currents.

References

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