Looking for a Diode in a 2-Dimensional Network

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1 Introduction

In this paper I chronicle my attempts to determine the presence and location of a diode in a square network of resistors, and to determine the values of those resistors. First I define a discretized Dirichlet to Neumann map for a resistor network that contains a diode. I then give an algorithm for computing some of the resistors in a network containing a diode, using the Dirichlet to Neumann map. Finally, I describe my attempts to find the location of the diode and remaining resistors.

2 The Dirichlet to Neumann Map

It is shown, in Curtis and Morrow, that a matrice Λ can be constructed from a network of resistors, which maps voltages on the boundary of the network to currents on the boundary of the network. Λ is an NxN matrice, where N is the number of exterior nodes in the network Γ. The *ith* column of Λ contains the currents into Γ resulting from placing a voltage of 1 at the *ith* exterior node and a voltage of 0 at every other node. Thus by superposition :

$$\Lambda x = b$$

maps external voltages x onto external currents b.

3 A Dirichlet to Nuemann map with a Diode

A *diode* is a nonlinear electrical device which only allows current to flow through it in one direction. An *ideal diode* $\underline{d}$ allows current to flow between its base and its tip if and only if the potential at its base, V_b is greater than the potential at its tip, V_t. If $V_t \geq V_b$, $\underline{d}$ becomes an open circuit, and the current flow is zero. In this paper, when I say diode I shall always mean an ideal diode [1].

[1] In actuality, a not-so-ideal diode generally becomes a short circuit when the voltage across it is greater than .6 volts, rather than 0 volts.

Furthermore, any diode present in a resistor network shall be assumed to be between two interior nodes of the network (and not between an interior node and an exterior node) and in series with a resistor between the same two nodes. Lastly, a diode with current flowing through it is said to be *on*, whereas a diode without current flowing through it is said to be *off*.

```
Here is the electrical symbol for a diode.  The arrow represents
direction of flow.  Current flows from the base to the tip only.

                          (base) ----|>|-------\/\/\/----(tip)
                                   (diode)  (resistor)
```

For a network Γ^* that contains an ideal diode, I define a new mapping, Λ^*. This Λ^* is constructed in the same way as Λ, i.e.: every *ith* column of Λ^* contains the external currents into Γ^* that result from placing a voltage of 1 at the *ith* exterior node and a voltage of 0 everywhere else. Λ^* does not, however, map external voltages to external currents, except in special cases which I shall explain below.

I define, for every internal resistor γ in Γ, a Λ^{oc}, which is the Λ matrice that would result if γ were broken; i.e. if it was an open circuit with infinite resistivity.

For each *ith* column of Λ^*, that column shall be the same as the *ith* column of Λ, (where Λ is assumed to come from a resistor network Γ identical to Γ^* except for the presence of the diode) *if and only if* the diode is *on*. If the diode is *off* then no current can flow directly from the interior node at the diode's tip to the interior node at the diode's base; it is an open circuit. Hence the *ith* column of Λ^*, is identical to the ith column of Λ^{oc} (for the resistor γ in series with the diode $\underline{d}$) whenever the diode is *off*.

REMARK 1: A diode must be either on or off. The general method of solving networks containing diodes is called the *method of assumed states*. With one diode in the network, we first assume the diode is on (don't forget that a diode has NO resistance of its own), hence current may flow through it, and through the resistor in series with it. We then solve for V_b and V_t. If $V_b < V_t$ then the initial guess was wrong and the diode is off. Solving once more for V_b and V_t, this time with an open circuit between them, we indeed find that V_b is still less than V_t. The reason for this becomes clear when one imagines a short circuit between the two nodes, making their respective voltages equal. Now increase the resistance between the nodes until it is infinite (open circuit). Clearly the voltage difference between the nodes must increase continually as the resistance increases. It follows that if $V_b < V_t$ with some resistance σ between them, then for an infinite resistance between them, the difference between them must be even greater.

4 The Inverse Problem

The inverse problem that I attempted to solve, with only partial success, was to recover the values of all the resistors, and the location of the diode, using only the information contained in Λ^*. First I shall give a method for determining which columns of Λ^* come from Λ, and which ones come from Λ^{oc}. I shall then present certain facts I was able to determine about all Λ^*s, and a method for recovering some (but not all) of the resistors in Λ^*. Lastly I shall show give a method for finding all the resistors and the position of the diode, if more boundary measurements are allowed.

I define an exterior node of a resistor network that contains a diode to be *off* if placing a voltage of 1 at that exterior node, and a voltage of 0 at every other exterior node, causes the potential at the tip of the diode to be higher than the potential at the base of the diode, which causes the diode to act like an open circuit. Similarly, an exterior node is *on* if a voltage of 1 at that exterior node and 0 elsewhere causes the diode itself to be *on* (act like a short circuit).

Throughout this paper I shall refer to both columns of Λ^* matrices and boundary nodes of Γ^* resistor networks as being *on* or *off*. There is no distinction; each Λ column corresponds to one boundary node and vice versa.

The method to determine which columns of Λ^* are on and which are off follows from Ohm's law:

$$I = VG$$

where V stands for voltage, I for current, and G for conductivity. Assuming a voltage of 1 at node i and a voltage of 0 at every other node, the sum of the currents flowing through the the network is equal to the current flowing into the network at exterior node i.[2] This in turn equals VG equals G_i, the equivalent conductance between exterior node i and ground. I am not interested in calculating G_i (it would be a nasty job); suffice it to say that adding a resistor anywhere inside the network will cause G_i to increase. One can see this more easily using an argument like that used in REMARK 1:

If the network contain only 1 resistor, it would certainly have a lower conductivity than if it contained 2 resistors in series. (Recall that conductivities in series add.) On the other hand, if the network were replaced with a sheet of copper (many many resistors in series), its conductivity would be much much higher. Adding one resistor to a network must increase G_i, as it give the current another path to follow. Therefore, the diagonal entries of a Λ^{oc} must always be less than the corresponding entries in Λ. Λ has a higher conductivity, and hence more current flows. We also know that in each column of Λ^* (or any Λ) the sum of the absolute values of the off-diagonal entries must equal the on-diagonal entry. Lastly, recall that Λ and Λ^{oc}, but not $Lambda^*$, are symmetric. From all this we obtain this rule:

[2]remember that all the other other nodes will have negative current flow. Also, by Kirchoff's Current Law, the sum of the currents into a network must be 0.

$$\forall \text{ exterior node } i, \text{ IF (the } ith \text{ column)}$$

$$\sum_{\substack{j \neq i \\ j \epsilon ext.nodes}} |\Lambda^*(j,i)| < \sum_{\substack{j \neq i \\ j \epsilon ext.nodes}} |\Lambda^*(i,j)|$$

(the *ith* row) Then node i is *off*. Otherwise it is *on*.

4.1 Some Facts about $\Lambda*$

1. For any given Λ^*, at least one boundary node must be *on* and one must be *off*.

   Suppose all the the boundary nodes are on. Starting at node 1, start placing a voltage of 1 at each boundary node of the network, Γ^*. By superposition the solutions must add, so the current through the diode must increase every time another voltage of 1 is added. When there is a voltage of 1 at *every* exterior node, there will be a very non-zero current running through the diode. But since voltage is a is a γ harmonic function, and the exterior values are all 1, the interior voltages must all be 1 also, and no current can flow. This is a contradiction. A similar argument holds for the case when all the boundary nodes are off.

2. The two boundary nodes occuring around a corner of Γ^* must be either both *on* or both *off*.

   This follows from the fact that currents and voltages from one corner node must differ from those of the other corner node by a positive constant. If one was *off* and the other *on* then by superposition current flowing from corner node A would have to choose a noticeably different route through Γ^* than current from corner node B, which makes no sense.

3. *CONJECTURE:* All the *ons* and *offs* in the boundary of a given Γ^* must come in two simply connected, complementary sets: a set of *ons*, and a set of *offs*.

   I never encountered a pattern of ons and offs that would violate this conjecture, but a conjecture it remains.

4.2 A Method to Compute *Some* of the Resistors from Λ^*

I shall number the corners of a square network from 0 to 3, proceeding clockwise and starting in the upper right hand corner. Let $corner_{j,i,a}$ be the boundary node i nodes to the left of corner j.(The numbering shall start with 1 at the corner.) $Corner_{j,i,b}$ shall be the boundary node i nodes to the right. I shall define the *ith* ladder of corner j as those resistors within the trapezoid delimited by $corner_{j,i,a}, corner_{j,i+1,a}, corner_{j,i,b} and corner_{j,i+1,b}$. (see picture).

```
(corner 3)    f  f  f  f  f  f  (corner 0)
              |  |  |  |  #  *
           f---------------#--*-f
              |  |  |  |  |  #
           f------------------#-f
              |  |  |  |  |  |
           n--------------------n
              |  |  |  |  |  |
           n--------------------n
              |  |  |  |  |  |
           n--------------------n
              |  |  |  |  |  |
           n--------------------n
              |  |  |  |  |  |
(corner 2)    n  n  n  n  n  n   (corner 1)
```

```
This is a typical pattern of
ons and offs.  The ons are
 represented by n's, the offs
by f's.
 In this picture, *s are on
the first ladder of corner 0.
 #'s are on the second.
```

The algorithm given by Curtis and Morrow for computing a given $ladder_{j,i}$ requires knowledge of only those columns of Λ corresponding to the nodes from $corner_{j,i,a}$ to $corner_{j,i,b}$, inclusive. IF all those equivalent nodes in Λ^* are all *on*, it is then possible to compute the appropriate ladder of resistors as if the network did not contain a diode at all. This is also true for ladders bounded entirely by *offs*, *except* that if the diode is hidden within one of the ladders to be computed, it will cause wildly inaccurate answers. (This corresponds to attempting to compute the resistors of a network that contains a broken resistor.[3])

The following diagram shows which resistors could be recovered from Λ^* for the Γ^* network pictured above. Those resistors which cannot be found are marked with *s. Those which can be found only if the diode is somewhere among the stars are marked with hats. (^s)

[3]see *Locating Faulty Resistors in a Network*, Hudelson

```
   f  f  f  f  f  f
   |  |  *  *  |  |
f---^--*--*--*--^----f       * marks irrecoverable resistor
   ^  *  *  *  *  ^          ^ marks possibly recoverable resistor
f---*--*--*--*--*----f       (recoverable if the diode is not on
   *  *  *  *  *  *           that ladder)
n---*--*--*--*--*----n
   |  *  *  *  *  |
n------*--*--*-------n
   |  |  *  *  |  |
n--------------------n
   |  |  |  |  |  |
n--------------------n
   |  |  |  |  |  |
   n  n  n  n  n  n
```

4.3 Finding All the Resistors and the Diode

I define Λ^{**} to be the matrice obtained by putting a voltage of *negative* one, one at a time at each exterior node of Γ^*. Once again, the *ith* column of Λ^{**} contains the currents into Γ^* around the boundary obtained by putting a voltage of -1 at the *ith* boundary node, and 0s at all the other boundary nodes.

For every *ith* column in Λ^*, if that column is *on* (current flows through the diode when the voltage at 1 at that boundary node and 0 elsewhere), then the corresponding column of Λ^{**} must be *off*, and vice versa. The reason comes from superposition: if the network did not have a diode, adding the two columns together must result in zero current flow everywhere. Hence current in Λ_i^{**} must be in the opposite direction as current in Λ_i^*, over every resistor. (Of course the diode will only allow it to flow in one direction across it, but the voltage drop across it must still change sign from Λ^* to Λ^{**}.) Thus it is possible to reconstruct the original Λ and Λ^{oc}.

For each *ith* column in Λ^*, if it is *on* it becomes the *ith* column of Λ and Λ_i^{**} becomes the *ith* column of Λ^{oc}. If it is *off* it becomes the *ith* column of Λ^{oc} and Λ_i^{**} becomes the *ith* column of Λ.

Having found Λ and Λ^{oc}, we can first use Curtis and Morrow's method[4] to find the resistors. We can then use Hudelson's method[5] to find the location of the diode, since in Λ^{oc} the diode is reduced to an open circuit.

[4] *Determining the Resistors in a Network*
[5] *Locating Faulty Resistors in a Network*

4.4 Two Diodes Between the Same 2 Interior Nodes

One diode in a network can be treated as a degenerate case of the picture below. When the $V_A > V_B$,diode 1 is *on*, diode 2 is *off*, and current flows through resistor 1. If $V_B > V_A$, diode 2 is *on*, diode 1 is *off*, and current flows through resistor 2. In the case we have until now dealt with, resistor 2 had infinite resitivity (or 0 conductivity) and is really an open circuit. Resistor 2 can, however, have a finite resistance without any harm acruing to the methods described above. The definition of *on* and *off* must change somewhat; *off* now means current flows through the resistor with the higher resistivity, *on* through the lower. Indeed, if resistor 1 = resistor 2 the diodes cannot be detected.

```
           (resistor 1)  (diode 1)
           |--/\/\/\/-----|>|------|
           |                       |
A---------|                       |-----------B
           |----|<|----/\/\/\/-----|
           (diode 2)     (resistor 2)
```

The same number of resistors can be found from Λ^* as before (with the added bonus that the diode's presence among off columns will no longer cause errors as it did when it was a short circuit), and when using Λ^{**}, Hudelson's methods no longer need to be used, as there are no broken resistors. Λ and Λ^{oc} can both be given the inverse treatment; the results will be identical except for one resistor in each network, which will be different.

5 Examples

I end with a few examples of the on off patterns generated in a 6x6 Γ^* network of constant conductivities. The dollar signs mark the location of the diode in each case, and stars mark those resistors which I could not recover from Λ^*.

```
   f  f  n  n  n  n
   |  *  |  |  |  |
f----*-*-------------n
   $  *  *  |  |  |
n---*--*--*----------n
   |  *  *  *  |  |
n------*--*----------n
   |  |  *  |  |  |
n--------------------n
   |  |  |  |  |  |
n--------------------n
   |  |  |  |  |  |
n--------------------n
   |  |  |  |  |  |
   n  n  n  n  n  n
```

```
   n  f  f  f  n  n
   |  *  *  *  |  |
n-----*--*--*--------n
   |  *  $  *  *  |
n-----*--*--*--------n
   |  |  *  *  |  |
n--------*-----------n
   |  |  |  |  |  |
n--------------------n
   |  |  |  |  |  |
n--------------------n
   |  |  |  |  |  |
n--------------------n
   |  |  |  |  |  |
   n  n  n  n  n  n
```

```
  f  f  f  f  f  f
  |  |  *  *  *  |
f-----*--*--*--*-----f
  |  *  *  *  *  *
f--*--*--*--*--*-----n
  *  $  *  *  *  |
n--*--*--*--*--------n
  |  *  *  *  |  |
n-----*--*-----------n
  |  |  *  |  |  |
n--------------------n
  |  |  |  |  |  |
n--------------------n
  |  |  |  |  |  |
  n  n  n  n  n  n
```

```
   f  f  f  f  f  f
   |  |  |  |  |  |
f---------*----------f
   |  |  *  *  |  |
f------*--*--*-------f
   |  *  *  *  *  |
f---*--*--*--*--*----f
   $  *  *  *  *  *
n---*--*--*--*--*----n
   |  *  *  *  *  |
n------*--*--*-------n
   |  |  *  *  |  |
n---------*----------n
   |  |  |  |  |  |
   n  n  n  n  n  n
```

```
   f  f  f  f  f  f
   |  |  |  |  |  |
f---------*----------f
   |  |  *  *  |  |
f------*--*--*-------f
   |  *  *  *  *  |
f---*--*--*--*--*----f
   *  *  $  *  *  *
n---*--*--*--*--*----n
   |  *  *  *  *  |
n------*--*--*-------n
   |  |  *  *  |  |
n---------*----------n
   |  |  |  |  |  |
   n  n  n  n  n  n
```