Improving the condition of the inverse problem

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1 The network.

In this paper we shall consider square networks of resistors whose nodes have integer coordinates. We will refer to such a system as an $n \times n$ network, where n is the number of nodes across the top row. The figure below illustrates the case when $n = 5$.

In general, each network has n^2 interior nodes, $4n$ boudary nodes, and $2n(n+1)$ resistors. As shown we will number the boundary nodes from $-n$ to $3n$ (for convenience of notation later on) clockwise beginning at the lower left and number the corners from 1 to 4 as shown.

Each node p in the network has a potential $u(p)$, each resistor connecting adjacent nodes p and q has conductance $\gamma(pq) > 0$, and by Ohm's Law the current flow through each resistor is $I(pq) = \gamma(pq)(u(p) - u(q))$ if $u(p) \geq u(q)$ or similarly of $u(q) > u(p)$. Along the boundary current flowing into the network will be considered positive and current flowing out of the network will be considered negative.

We require also that the sum of the individual currents at each interior node is zero (Kirkhoff's Law), so that no such node is a source or sink of current. In symbols, if the q_i's are the neighbors of an interior node p, we require

$$\sum_{i=1}^{4} \gamma(pq_i)((u(p) - u(q_i)) = 0.$$

2 Dirichlet to Neumann map.

Given the conductance of each resistor and the potentials at the boundary, the "forward" problem is the determination of the potential at each interior node. This can be done by applying Kirkhoff's Law at each interior node, yielding n^2 equations in n^2 unknown potentials, which can be solved by Gaussian elimination.

Once a particular forward problem has been solved, we then may determine the current flow along the boundary from the given formula for current. The mapping from the space of boundary potentials to the space of boundary currents for a particular network is called the Dirichlet to Neumann map, denoted by Λ. We may find Λ as follows : For $j = 1,\ 2,\ ...,\ 4n$

let ϕ_j be the potential vector (in $4n$-space) consisting of 1 at boundary node j and zeros elsewhere. Let C_j be the solution of the forward problem using ϕ_j; then the C_j's are the columns of the Λ matrix.

The units of Λ are current per unit voltage, so that multiplying a potential vector by Λ yields a current vector. Curtis and Morrow have shown that Λ is symmetric, with positive entries on the (main) diagonal and negative entries elsewhere. Because no current is created or absorbed in the interior of the network, the sum of the entries of each column (row) of Λ is zero. The matrix also contains the so-called corner relations, a key property which will enable us to show that, given Λ, we may determine the conductance function γ of the network.

3 Corner relations.

Let us define the k^{th} diagonal at corner 1 as the set of $k+2$ nodes along a straight line from node $k+1$ to node $-k-1$. Curtis and Morrow have shown that it is always possible to obtain potentials of zero at all nodes on and below (above) any diagonal. Consider the following :

If we wish to set zero potentials along the k^{th} diagonal, it is enough to (1) set the potentials to zero at nodes $k+1,\ k+2,\ ...$ clockwise around to $-k-1$, and (2) set the current to zero at nodes $n+1,\ ...,\ n+k$. [We will see that we have a certain degree of freedom regarding this choice.] These conditions yield zero potentials and zero currents on and below the k^{th} diagonal, and we are free to impose any potentials we like at nodes $-k,\ -k+1,\ ...,\ k$.

Suppose then that we assume such a scheme along the first diagonal, and that we set the voltage to 1 at node -1. At this node some current I will flow into the network, which must flow out at node 1, thus determining some voltage α there. In terms of the columns (C_j's) of Λ, we have now obtained the *first relation at corner 1,*

$$C_{-1} + \alpha C_1 = \begin{cases} I \textit{ at node } \text{-1}, \\ -I \textit{ at node } 1, \\ 0 \textit{ elsewhere.} \end{cases}$$

In general the k^{th} *relation at corner* 1 is obtained by imposing zero voltages along the k^{th} diagonal, setting voltages of 1 at node $-k$ and zeros at the nodes directly above, and reading off the voltages $\alpha_1,\ ...,\ \alpha_k$ across the top. The equations are

$$C_{-k} + \sum_{i=1}^{k} \alpha_i C_i = \begin{cases} I_j \ for \ j = -k, \ -k+1, \ ..., \ k, \\ 0 \ elsewhere \ (in \ particular \ for \ j = n+1, \ n+2, \ ..., \ 3n). \end{cases}$$

(Note that C_{-k} refers to the $(4n+1-k)^{th}$ column of Λ).

There are analogous relations at corners 2, 3, and 4. In regards to condition (2) above, Curtis and Morrow have shown that we may set any k currents from $n+1$ to $3n$ to zero *so long as all combinations of nodes where there exists a corner relation are avoided,* i.e. nodes $2n$ and $2n+1$, nodes $2n-1,\ 2n,\ 2n+1$, and $2n+2$, etc.

4 Inverse problem.

We now have stated enough properties of Λ to recover the conductance of the network by the direct or staircase method. It is sufficient to describe the process for one-fourth of the network (a "wedge", below), as the rest may be found by appropriate rearrangements of the network and Λ.

Let us define the k^{th} staircase as the set of k resistors in the wedge between the $(k-1)^{st}$ and k^{th} diagonals. Number the resistors $\gamma_1,\ \gamma_2,\ ...,\ \gamma_k$ towards the center of the network. In the first step, we set the first diagonal to zero and use the $(n+1)^{st}$ entry of the relation $C_{-1} + \alpha_1 C_1 = 0$ to find α_1. We can now read off the current I_{-1} from the relation, thus determining γ_1 since $I_{-1} = (1-0)\gamma_1$.

The recovery of the conductance of a wegde requires n steps. If we assume that $(k-1)$ steps have been completed, we solve the k^{th} step as follows : Impose zero potentials along the k^{th} diagonal, and impose suitable conditions so that the k^{th} relation holds. We set currents $n+1,\ ...,\ n+k$ to zero; this corresponds to using the $(n+1)^{st},\ ...,\ (n+k)^{th}$ entries of the k^{th} relation at corner 1 in order to find $\alpha_1,\ ...,\ \alpha_k$ by Gaussian elimination. Since the α's are now known, we can read off the currents $I_{-1},\ I_{-2},\ ...,\ I_{-k+1}$ from the k^{th} relation. Now repeatedly apply Kirkhoff's Law on the interior of the wedge to find the voltages along the $(k-1)^{st}$ staircase. We set up k equations in $\gamma_1,\ ..., \gamma_k$ by applying Kirkhoff's Law at $(k-1)$ wedge nodes along diagonals $(k-1)$ and k, plus the equation $I_{-k} = (1-0)\gamma_1$. This completes step k.

5 Improvements.

Unfortunately the method just described is ill-conditioned. The main cause is that the computation of α's from the k^{th} corner relation becomes increasingly singular as k increases. For example, at the n^{th} step of the direct method, we use the relation $C_{-n} + \sum_1^n \alpha_i C_i = 0$, and solve for the α's by setting currents $n+1,\ \dots,\ 2n$ to 0. This system in, say, a 12×12 case typically has a condition number on the order of 10^{14}. The innacuracy of the α's in such a case results in γ-solutions nowhere near their actual values (frequently negative).

This problem can be remedied somewhat. As previously mentioned, at the k^{th} step we are free to impose k currents from the set $n+1,\ \dots,\ 3n$ to zero, so long as corner relation nodes are avoided. But columns $C_{n+1}, \dots, C_{2n}$ (which we have so far been choosing) of Λ are "nearly" linearly dependent in the sense that columns $C_{n+1}, \dots, C_{2n+1}$ are dependent. One possible scheme is to stay away from the relation at corner 3 as much as possible. In this "improved current" method, the condition number of solving for the α's can be lowered by dividing up the currents to be set to zero, putting half of them directly below corner 2 and half of them to the right of corner 4.

Another problem in the staircase method is the relative size of the α's. For example, in the tenth step of a 10×10 network (with conductances between 1 and 9), α_1 is several orders of magnitude larger then α_{10}. This can be improved somewhat by moving half of the α's just above the unit voltage at node $-k$ (replacing them with zeros) and leaving half of them α's to the left of node k. The "relation" used in solving for the α's in this case is

$$C_{-k} + \sum_{i=1}^{[k/2]} \alpha_i C_{-k+i} + \sum_{i=1+[k/2]}^{k} \alpha_i C_i = 0.$$

This is the "improved α" method (figure 5 with $n = 5$, $k = 4$), and when combined with the improved current method the condition numbers of the steps are brought down further. The result is that the conductance function γ is recovered more accurately.

6 Results.

There following are data for three networks of conductances between 1 and 9. The condition numbers here are the reciprocals of those that were returned by LINPACK double precision matrix-solver routines. For simplicity, data from the last few steps of the recovery of the first wedge are given, since the data from the other wedges are of like magnitudes.

8×8	direct	improved α	improved current	both
step 5	2.7×10^6	3.1×10^5	2.9×10^4	5.8×10^3
step 6	3.6×10^7	2.0×10^6	5.7×10^4	1.4×10^4
step 7	4.3×10^8	5.6×10^6	7.9×10^4	1.9×10^4
step 8	1.1×10^{10}	8.4×10^6	8.9×10^5	5.0×10^4

10×10	direct	improved α	improved current	both
step 7	3.8×10^9	1.6×10^8	1.5×10^6	3.8×10^5
step 8	3.8×10^{10}	4.0×10^7	2.0×10^6	1.5×10^5
step 9	5.3×10^{12}	3.4×10^9	2.5×10^6	7.5×10^5
step 10	2.9×10^{14}	3.7×10^9	2.9×10^6	2.6×10^5

12×12	direct	improved α	improved current	both
step 9	1.1×10^{13}	3.5×10^{10}	1.1×10^8	2.3×10^7
step 10	1.4×10^{14}	1.2×10^{11}	1.4×10^8	2.1×10^7
step 11	2.1×10^{15}	6.8×10^{11}	1.5×10^8	3.1×10^7
step 12	6.1×10^{16}	4.0×10^{12}	1.6×10^8	4.2×10^7

7 Further improvements.

The accuracy of the method still can be increased. Consider the fourth step of the inverse problem of a 5×5 network (we omit the currents for simplicity) :

As we know, in this step we are computing resistors γ_1, γ_2, γ_3, and γ_4 by using the fourth relation at corner 1. However, γ_1 could have been found much more accurately from the second relation at corner 4 (the voltages in brackets require the computation of only two α's).

Therefore a better implementation of solving the inverse problem would be to go by "levels": At level 1 we compute every possible first staircase (there are eight), at level 2 we would compute every possible second staircase, and so on until we reach level $1 + [n/2]$. At this point we would solve each step only for the resistors of the staircase that were presently unknown. For example, at the fourth level above we would only need to solve for γ_4, as we would know γ_1, γ_2, and γ_3 from previous steps. An implementation of this sort would greatly improve the number of digits of accuracy in the recovered resistors.