Comparison of Methods for Finding Resistors in a Network

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1 PRESENTATION OF GOAL

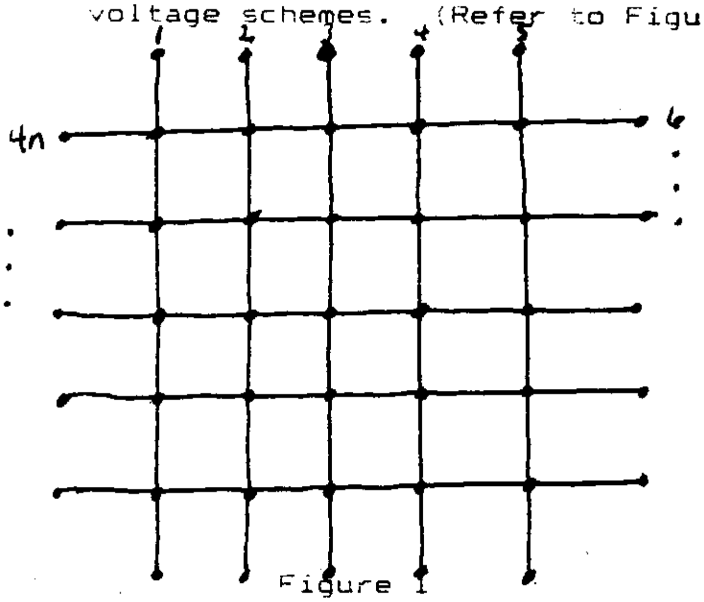
The goal attempted was to analyze and evaluate various algorithms utilizing available solvers in order to produce the most sensitive method for detecting one or more "peculiar" resistors among a majority of conductivities equal to 1 in a square network. Initially, I embarked to find the program which solved the Inverse Problem given boundary data, with the least maximum error for a square network consisting of resistors with conductivities all equal to 1. Also taken into account were the accuracies of each with respect to the dimension of the network. The basic algorithms were then altered from an exactly determined system to degrees of overdetermination.

2 REVIEW OF INVERSE PROBLEM

Given a square network of dimension n, for which there are n boundary nodes on a side, the voltages of the interior nodes may be readily determined from the boundary data and known conductivities. The current flowing into any boundary node is calculated using Ohm's law, which states that

$$I(p,q)=g(p,q)\times(u(p)-u(q))$$

where I(p,q) is a boundary current, p is a boundary node, q is the unique interior neighbor of p, g(p,q) is the conductivity of resistor (p,q), and u(p) and u(q) are the voltages at nodes p and q, respectively. Consider a voltage scheme as when one of the 4n boundary node potentials is set equal to 1 while all others are set equal to 0. The currents at each of the 4n boundary nodes are taken for each of 4n voltage schemes. The calculated currents are stored in a matrix Lambda. Each row of Lambda corresponds to a particular voltage scheme and each column contains the currents at a particular node with respect to each of the 4n voltage schemes. (Refer to Figure 1)



n=5

Lambda(1,2) is equal to the current at node 2 when node 1 equals 1 and all other boundary nodes equal zero.

Figure 1

Knowing the current information from Lambda allows for the calculation of the conductivities of all resistors in the network.

3 ORIGINAL METHOD

The control or original method for finding resistors in a network was taken from the 1988 documentation of Thad Edens entitled "Calculating the Resistors in a Network". This method was used as a standard of accuracy in comparison to a second method to be introduced later. The method involves first calculating the interior voltages and the currents of the resistors. From the calculated data, the conductivities are directly computed by solving Ohm's law for conductivity.

Consider a square resistor network and number the boundary nodes going clockwise from 1 to 4n.(Refer to Figure 2) Let the potential of node n, u(n), equal 1 and have the other 4n-1 boundary nodes have zero potentials. Find the current flowing from node n to to the node immediately south by multiplying the nth row of the matrix Lambda by the vector of boundary potentials. The nth entry of the solution vector will be the sought current value, I(n,p1). Assuming the conductivities of all resistors are initially known, use a manipulated version of Ohm's law to calculate u(p1). With the given and calculated data, the current I(p1,n+1) is determined. The conductivities of resistors (n,p1) and (p1,n+1) may be determined by solving Ohm's law for the conductivity value.

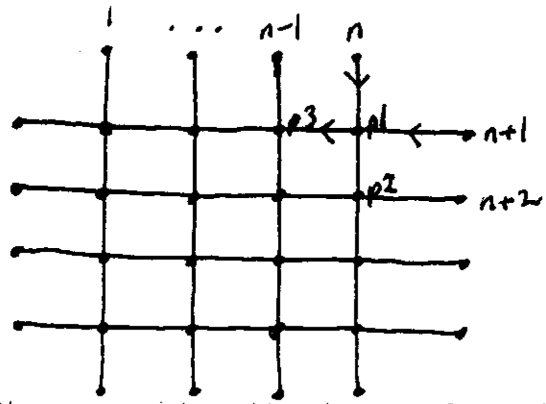


Figure 2

Now consider the band of resistors beginning with boundary node n-1. According to the voltage scheme where u(n-1) equals 1 and all other boundary nodes have voltage 0, re-determine the currents and interior voltages which occur in the band beginning with boundary node n. Begin calculating the band below n-1 by finding the boundary current I(n-1,p3) using Lambda, and the voltage u(p3) using Ohm's law. Since u(p1) has already been calculated and given g(p3,p1), it is possible to calculate the current into p1, I(p3,p1)=g(p3,p1)x(u(p1)-u(p3)). In determining I(p1,p2), it is known that the current flowing out of node p1 through resistor (p1,p2) is equal to the sum of the currents flowing into p1 through (n,p1), (p1,n+1), and (p3,p1). From the previous band, the currents through (n,p1) and (p1,n+1) into p1 are known. But I(p3,p1) was the calculated current flowing out of p1, so its sign must change to show the current flowing into p1. Therefore,

$$I(p1,p2)=I(n,p1)+I(p1,n+1)-I(p3,p1).$$

The final current in the band, I(p2,,n+2), is gotten by Ohm's law. Again calculate the resistors in the band below n-1 using the current and voltage information. Proceed in this way for the bands below nodes n-2 through 1 , always calculating the currents and interior voltages of the previous bands for each new voltage scheme.

4 ROW PROGRESSION METHOD

The method created to be tested against the original method will be referred to as the row progression method. It incorporates with Ohm's law the use of Kirkhoff's law:

$$u(p)\times(g(p,q1)+g(p,q2)+g(p,q3)+g(p,q4))=$$
$$u(q1)\times g(p,q1)+u(q2)\times g(p,q2)+u(q3)\times g(p,q3)+u(q4)\times g(p,q4)$$

(Refer to Figure 3)

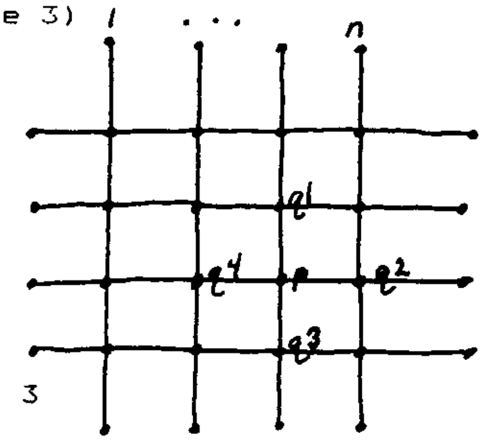


Figure 3

Begin by calculating the first two resistors at the top right-hand corner by using current information and assume the bands of resistors below boundary nodes n through k+1 have been previously calculated. In order to calculate the band below node k, consider the voltage scheme were u(k)=1 and u(i)=0 for all boundary nodes i not equal to k. Assume u is a harmonic function so that according to Curtis and Morrow, the diagonal below node k is such that all nodes on and below the diagonal have voltages equal to zero.(Refer to Figure 4) Then successively multiply rows k+1 through n of Lambda by the vector of 4n boundary potentials in order to obtain the top

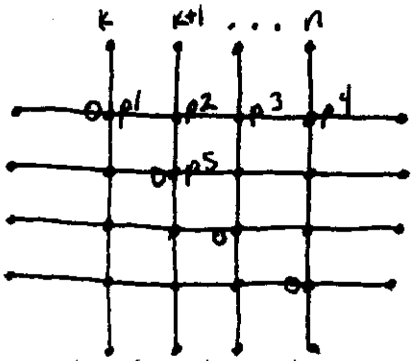


Figure 4

row of currents for boundary nodes including and to the east of k. Immediately, the conductivity g(k,p1) may be found by Ohm's law. The given conductivities along with the boundary current information enable the nonzero voltages u(p1), u(p2), u(p3), and u(p4) to be calculated using Ohm's law. Next use Kirkhoff's law with the newly obtained value of u(p2) to determine g(p1,p2). Use Kirkhoff's about node p2 to find g(p2,p5). Again, use the previously calculated bands of resistors and the second row voltages to determine the third row voltages by executing Kirkhoff's law about about each nonzero second row node. Continue finding the resistors below node k by traveling east then south, then calculating the next row of voltages.

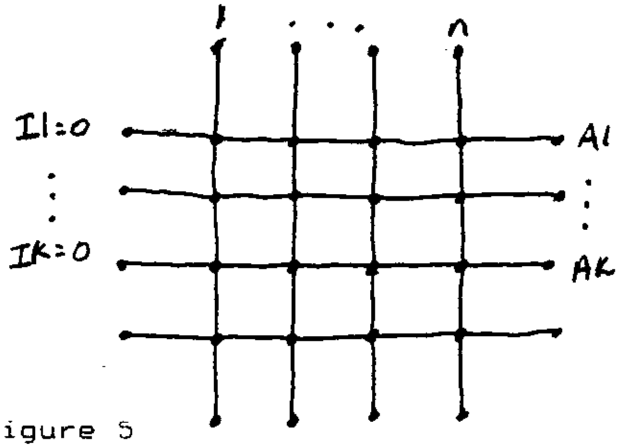


Figure 5

Consider the potential 1 as being applied to one of the nodes 1 through n. According to Professors Curtis and Morrow, the nonzero voltages on the right-hand side are determined by the boundary currents on the left-hand side provided+ that the left-hand side voltages are uniform. Number the right-hand side nonzero nodes as A1 through Ak. Nonzero values up to Ai require i currents on the left-hand side to equal 0.(Refer to Figure 5) Proportions throughout the Lambda matrix allow for the following equality when k left-hand side currents equal 0.

$$0 = 1 \times \text{Lambda}(i,i) + (A1 \times \text{Lambda}(i,1) + A2 \times \text{Lambda}(i,A2) + \ldots$$
$$\ldots + Ak \times \text{Lambda}(i,k)) \text{ for } i=1,\ldots,k.$$

where Lambda(x,y) is the entry of the Lambda matrix corresponding to the current at node y with respect to the voltage scheme where boundary node u(x)=1. There are k of the above equations which allow for solving for k unknown values of A1,...,Ak

In an attempt to improve results of the original and row progression methods, two other systems of equations were solved using the Least Squares solver. The first system set all n left-hand side currents equal to 0 so that there were always n equations for less than or equal to n unknowns. The second set the left-hand side and bottom row boundary currents equal to 0 so that 2n equations were always solved for 1 through n unknowns.

5 VARIATIONS AND IMPROVEMENTS

The results of the row progression method for conductivities were progressively less accurate than the original method as the network dimension increased. Resistor bands were calculated by the row progression method beginning with the upper right-hand corner until half of all resistors were determined. Bands were then calculated beginning in the lower left-hand corner and working towards the opposite corner. Since the last vertical resistors calculated in the last band by each cycle were the worst-conditioned, an improvement may be sought by re-calculating the resistor bands beginning at the remaining two opposite corners. Each resistor may thus determined twice and the more accurate value for each saved.

Initially, the original method was flawed in that the diagonal node voltages below the boundary node k, where u(k)=1, were calculated as having values approximate to 0 when they should have exactly equaled 0. Conductivities produced by the original method were bettered in an improved original method by automatically setting the diagonal node voltages below the kth boundary node to 0 when calculating the interior currents.

6 RESULTS

Nine programs in all were constructed: the original method- exactly determined, slightly overdetermined, fully overdetermined; the improved original method with diagonal entries set to 0- exactly determined, slightly overdetermined, fully overdetermined; and the row progression method- exactly determined, slightly overdetermined, fully overdetermined.

In order to truly test the sensitivity and accuracy of the two methods, conductivities of 5 were assigned to the worst-conditioned resistors and then to the center most resistors in the network. The maximum dimension for which the 5-resistors were detected by each method indicated the sensitivity of each.

7 NUMERICAL RESULTS

Maximum relative error with all conductivities equal to 1

Status: Exactly determined

n	Original	Improved Original
6	$8.3643092452235\times10^{-11}$	$7.1047168148652\times10^{-11}$
8	$8.0228286902795\times10^{-8}$	$6.0689800740477\times10^{-8}$
10	$3.7706357571388\times10^{-5}$	$2.8273036663196\times10^{-5}$
12	$6.6482693532295\times10^{-2}$	$2.7623141861161\times10^{-2}$
15	244.03701976892	221.20994834160

	Row Progression
6	$1.0989742449397\times10^{-10}$
8	$4.8411956332650\times10^{-7}$
10	$3.6717458527635\times10^{-4}$
12	0.42393632328757
15	3601263.4181223

Status: Slightly overdetermined

n	Original	Improved Original
6	$9.6606500576968\times10^{-11}$	$8.1779916172309\times10^{-11}$
8	$7.1069471641039\times10^{-8}$	$5.2673198025488\times10^{-8}$
10	$2.1598503319842\times10^{-5}$	$1.6698512846913\times10^{-5}$
12	$4.2268245857318\times10^{-2}$	$3.2118601818223\times10^{-2}$
15	5237.7522524017	38.512402430473

	Row Progression
6	$3.8610159514008\times10^{-10}$
8	$6.9646310696392\times10^{-7}$
10	$5.4693865735789\times10^{-4}$
12	0.20048685017164
15	424786.99511053

Status: Fully overdetermined

	Original	Improved Original
6	$2.0864932004372\times10^{-10}$	$1.6753887166487\times10^{-10}$
8	$1.4925366653706\times10^{-7}$	$1.1394133969667\times10^{-7}$
10	$5.3065528070162\times10^{-5}$	$3.9580134225115\times10^{-5}$
12	$3.3133409840518\times10^{-2}$	$2.5124595110911\times10^{-2}$
15	1310.7301273084×10	16.304281929736

	Row Progression
6	$5.1770365772086\times10^{-11}$
8	$2.2905016106023\times10^{-7}$
10	$7.3956846801515\times10^{-4}$
12	0.70807518886551
15	35609.904778790

Detecting 5-resistor

Status: Fully overdetermined

Improved Original	Row Progression
n=13	n=12
cond. no.=0.76229498001737	cond. no.=1.1972442899967