ON A CHARACTERIZATION OF THE KERNEL OF THE DIRICHLET-TO-NEUMANN MAP FOR A PLANAR REGION

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Abstract. We will show that the Dirichlet-to-Neumann map Λ for the electrical conductivity equation on a simply connected plane region has an *alternating property*, which may be considered as a generalized maximum principle. Using this property, we will prove that the kernel, K, of Λ satisfies a set of inequalities of the form $(-1)^{\frac{n(n+1)}{2}} \det K(x_i, y_j) > 0$. We will show that these inequalities imply Hopf's lemma for the conductivity equation. We will also show that these inequalities imply the alternating property of a kernel.

1. Introduction

In this paper we will derive some properties of the Dirichlet-to-Neumann map for the electrical conductivity equation in $\mathbf{R}^2$. These properties are analogs of properties which characterize the Dirichlet-to-Neumann maps for electrical networks (see [1] and [2]). We recall some definitions. Let Ω be a relatively compact, simply connected open set in $\mathbf{R}^2$ with C^2 boundary. Let $\gamma(p) > 0$ be a C^2 function on $\overline{\Omega}$. Let f be a function defined on $\partial\Omega$. Then there is a unique function u, defined on $\overline{\Omega}$, such that

$$\nabla(\gamma \nabla u) = 0 \tag{1.1}$$

and $u(p) = f(p)$, for $p \in \partial\Omega$. (Equation (1.1) is the electrical conductivity equation and a function, u, that satisfies (1.1) is called a $\underline{\gamma\text{-harmonic function}}$.) Let $\frac{\partial u}{\partial n}(p)$ be the directional derivative of u in the direction of the outward pointing unit normal n at the point $p \in \partial\Omega$. Then the Dirichlet-to-Neumann map, Λ, is defined by the formula

$$\Lambda f(p) = \gamma(p) \frac{\partial u}{\partial n}(p). \tag{1.2}$$

The domain of Λ may be taken to be $H^{\frac{1}{2}}(\partial\Omega)$ and the image is in $H^{-\frac{1}{2}}(\partial\Omega)$. Λ is a pseudo-differential operator of order 1 and as such has a kernel,

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$K(x,y)$, defined as a distribution on $\partial\Omega \times \partial\Omega$. The kernel gives a representation of Λ by the formula

$$\Lambda f(x) = \int_{\partial\Omega} K(x,y)f(y)dy, \tag{1.3}$$

where x and y are arc length coordinates on $\partial\Omega$. For the pseudo-differential operator Λ, the kernel K is a symmetric function, $K(x,y) = K(y,x)$, and for a fixed $x \in \partial\Omega$, $\lim_{y\to x} |K(x,y)| = \infty$. More precisely,

$$K(x,y) = \frac{k(x,y)}{|x-y|^2} + D(x,y), \tag{1.4}$$

where k is continuous on $\partial\Omega \times \partial\Omega$, $k(x,y) = k(y,x)$, $k(x,x) \neq 0$, and D is a distribution supported on $\Delta = \{(x,x) : x \in \partial\Omega\}$. (In this formula, $|x-y|$ is the separation in arc length of points with arc length coordinates x and y and the continuous term in this expansion has been incorportated into the term $\frac{k(x,y)}{|x-y|^2}$.) If $x \notin \text{supp}(f)$, then the integral is an ordinary integral and there are no convergence questions. Since we will be interested in the behaviour of $K(x,y)$ for $x \neq y$ we will ignore D and will pretend that $K(x,y) = \frac{k(x,y)}{|x-y|^2}$. The expansion (1.4) follows from Lemma 3.7 of [6] or Theorem 0.1 in [7]. The boundary, $\partial\Omega$, is a Jordan curve and hence is homeomorphic to a circle. Pick an orientation on $\partial\Omega$. We say that $(x_1, \ldots, x_n; y_1, \cdots, y_n)$ is a circular pair if there are points $p, q \in \partial\Omega$ which divide $\partial\Omega$ into two connected components, A, B such that $\{x_1, \ldots, x_n\} \subset A$, $\{y_1, \ldots, y_n\} \subset B$, and $x_1, \ldots, x_n, y_1, \ldots, y_n$ are in circular order on $\partial\Omega$. (Note that this definition is modified from the definition in [2].) The main theorem of this paper is the following theorem, which we prove to be equivalent to the alternating property stated in section 2.

Theorem 1.1. *Let $(x_1, \ldots, x_n; y_1, \cdots, y_n)$ be a circular pair on $\partial\Omega$. Let $L = (l_{ij})$ be the $n \times n$ matrix with entries defined by $l_{ij} = K(x_i, y_j)$. Then*

$$(-1)^{\frac{n(n+1)}{2}} \det(L) > 0 \tag{1.5}$$

We consider this to be a generalization of a result in [2]. We will see how it implies the classical Hopf lemma for the conductivity equation in dimension 2.

2. The Alternating Property

We first restate and prove a result of [2]. Suppose that $\partial\Omega = I \cup J$, where I and J are disjoint connected arcs. Then we have the following theorem.

Theorem 2.1. *Let f be a smooth function on $\partial\Omega$ such that $f = 0$ on I. Suppose there is a sequence of points $\{p_1, \ldots, p_n\} \subset I$ in circular order such that*

$$(-1)^{i+1}\Lambda f(p_i) > 0. \tag{2.1}$$

Then there is a sequence of points $\{q_1, \ldots, q_n\} \subset J$ *in circular order such that*

$$(-1)^n \Lambda f(p_i) f(q_i) > 0. \tag{2.2}$$

Proof. Equation (2.2) is equivalent to

$$\Lambda f(p_i) f(q_{n+1-i}) < 0. \tag{2.3}$$

We first describe how to pick the point q_n. Let u be the solution of (1.1) such that $u = f$ on $\partial\Omega$. By (2.1) $\frac{\partial u}{\partial n}(p_1) > 0$. Hence there is a small open line segment, α, such that $\alpha \subset \Omega$, p_1 is one end of α and $u < 0$ on α. Let W be the connected component of $\{z \in \Omega \ : \ u(z) < 0\}$ that contains α. Suppose that $\overline{W} \cap J = \emptyset$. Then $u = 0$ on ∂W. But this contradicts the maximum principle since $u < 0$ in W and $W \neq \emptyset$. Thus $\overline{W} \cap J \neq \emptyset$. Now $u = 0$ at every point of ∂W that is in Ω. Using the maximum principle again we see that there is a $q_n \in \overline{W} \cap J$ such that $f(q_n) < 0$ and there is an open line segment $\beta \subset W$ such that q_n is an end point of β. Now we can connect the ends of α and β that are inside W by a smooth curve in W. Hence there is a smooth curve C_1 such that C_1 is diffeomorphic to a line segment, has end points points p_1 and q_n, and $C_1 - p_1 - q_n \subset W$. Then $u(z) < 0$ for all $z \in C_1 - p_1$. We can repeat this argument to produce curves C_j such that C_j joins p_j to a point $q_{n+1-j} \in J$, $C_j - p_j - q_{n+1-j} \subset \Omega$, and $(-1)^j u(z) < 0$ for all $z \in C_j - p_j$. These curves cannot intersect and by the Jordan curve theorem the points $p_1, \ldots, p_n, q_1, \ldots, q_n$ must be in circular order on $\partial\Omega$. It is easy to see that these points satisfy (2.3). ☐

We have referred to this property as the alternating property. Elsewhere ([5]) a similar property has been called the variation diminishing property. See also section 6 of this paper.

3. The Weak Inequality

We first prove the weaker statement:

Theorem 3.1. *Let* $(x_1, \ldots, x_n; y_1, \cdots, y_n)$ *be a circular pair on* $\partial\Omega$. *Let* $L = (l_{ij})$ *be the* $n \times n$ *matrix with entries defined by* $l_{ij} = K(x_i, y_j)$. *Then*

$$(-1)^{\frac{n(n+1)}{2}} \det(L) \geq 0 \tag{3.1}$$

Proof. The proof is by induction on n. We first consider $n = 1$. The proof goes by contradiction. Suppose that there are points $p, q \in \partial\Omega$ with $p \neq q$ and $K(p, q) > 0$. Then there is an $\epsilon > 0$ such that $p \notin D_\epsilon = \{y : |y - q| < \epsilon\}$ and $K(p, y) > 0$ for $y \in D_\epsilon$. Let $f(y)$ be a continuous function on $\partial\Omega$ such that $\text{supp}(f) \subset D_\epsilon = \{y : |y - q| < \epsilon\}$, $f(q) > 0$, and $f(s) \geq 0$ for all $s \in \partial\Omega$. Then

$$\gamma(p) \frac{\partial u}{\partial n}(p) = \Lambda f(p) = \int_{D_\epsilon} K(p, y) f(y) dy > 0,$$

where u satisfies (1.1) and $u(s) = f(s),\ s \in \partial\Omega$ But then there must be a point z near p in Ω such that $u(z) < 0$. This contradicts the maximum principle.

Next we assume that the result is true for all $(n-1) \times (n-1)$ matrices and prove that it is true for $n \times n$ matrices. If the result is not true, then we have a circular pair $(x_1, \ldots, x_n; y_1, \cdots, y_n)$ such that

$$(-1)^{\frac{n(n+1)}{2}} \det(L) < 0. \tag{3.2}$$

Consider the matrix L^{-1} with entries (h_{ij}). Then

$$h_{ij} = (-1)^{i+j} \frac{\det(L_{ij})}{\det(L)}, \tag{3.3}$$

where L_{ij} is the (i,j) minor of L. By induction, (3.2), and (3.3)

$$(-1)^{i+j+\frac{n(n-1)}{2}+\frac{n(n+1)}{2}+1} h_{ij} = (-1)^{i+j+n+1} h_{ij} \geq 0. \tag{3.4}$$

Since L is nonsingular, for fixed i there must be some j for which

$$(-1)^{i+j+n+1} h_{ij} > 0. \tag{3.5}$$

Now let $w = [1, -1, 1, \ldots, (-1)^{n+1}]^T$ be an n-vector with alternating signs. Let $z = L^{-1} w$. Then using (3.4) and (3.5) it is easy to verify that

$$(-1)^{i+n} z_i > 0. \tag{3.6}$$

To summarize, we have a vector z such that

$$(-1)^{i+1} = w_i = \sum_{j=1}^{n} K(x_i, y_j) z_j \tag{3.7}$$

and

$$(-1)^{n+1} z_i w_i > 0. \tag{3.8}$$

Now, choose small intervals D_j around the points y_j such that the D_j are disjoint and do not contain any of the points x_i. Choose the D_j so small that

$$|K(x_i, y) - K(x_i, y_j)| < \epsilon,\ y \in D_j,\ i = 1, \ldots, n. \tag{3.9}$$

Also choose functions f_j such that

$$\operatorname{supp}(f_j) \subset D_j,\ z_j f_j(y) \geq 0, \text{ and } \int_{D_j} f_j = z_j. \tag{3.10}$$

Let $f = \sum f_j$. Then

$$
\begin{aligned}
|\Lambda f(x_i) - w_i| &= |\int_{\partial\Omega} K(x_i, y) f(y) dy - \sum_{j=1}^{n} K(x_i, y_j) z_j| \\
&= |\int_{\partial\Omega} (K(x_i, y) - K(x_i, y_j)) f(y) dy| \qquad (3.11)\\
&\leq \epsilon \sum_{j=1}^{n} |z_i|.
\end{aligned}
$$

Thus we conclude that for ϵ small enough $\Lambda f(x_i)$ has the same sign as w_i. By the alternating property, there would have to be a set of n points t_i in circular order such that

$$(-1)^n w_i f(t_i) > 0. \tag{3.12}$$

For such a set of points we would have to have $t_i \in D_i$ and hence $f(t_i)$ would have the same signs as z_i. This contradicts (3.8).

□

4. The Strong Inequality

We now prove Theorem (1.1). We consider the cases $n = 1$ and $n > 1$ separately. Let us assume the arc length of $\partial\Omega$ is S and that points on $\partial\Omega$ are parametrized by the numbers in the interval $[0, S)$. When $n = 1$, suppose there is a pair of points x_1, y_1 with $0 \leq x_1 < y_1$ and $K(x_1, y_1) = 0$. By (1.4) there is no sequence of points z_j such that $x_1 < z_j < y_1$, $\lim_{j\to\infty} z_j = x_1$, and $\lim_{j\to\infty} K(x_1, z_j) = 0$. Hence there is a point η_2 with $x_1 < \eta_2 < y_1$ such that

$$K(x_1, \eta_2) = 0, \text{ and } K(x_1, \eta) < 0, \text{ for } x_1 < \eta < \eta_2.$$

Let x be any number such that $x_1 < x < \eta_2$ and choose η_1 so that $x < \eta_1 < \eta_2$. Then $(x_1, x; \eta_1, \eta_2)$ is a circular pair and hence

$$\begin{vmatrix} K(x_1, \eta_1) & K(x_1, \eta_2) \\ K(x, \eta_1) & K(x, \eta_2) \end{vmatrix} \leq 0. \tag{4.1}$$

Since

$$K(x_1, \eta_2) = 0, \; K(x, \eta_2) \leq 0, \text{ and } K(x_1, \eta_1) < 0,$$

it follows that

$$K(x, \eta_2) = 0. \tag{4.2}$$

This shows that for *all* x, with $x_1 < x < \eta_2$, $K(x, \eta_2) = 0$. Hence we get the contradiction that $\lim_{x\to\eta_2} K(x, \eta_2) = 0$.

The proof for $n > 1$, makes use of the following result in [2]. It was later pointed out to us that Charles Dodgson (Lewis Carroll) used a version of this identity in [4]. Let $(x_1, \ldots, x_n; y_1, \ldots, y_n)$ be a circular pair. We assume that the coordinates on $\partial\Omega$ are chosen so that $0 \leq x_1 < \cdots < x_n < y_1 <$

$\cdots < y_n < S$. Let L be the matrix with i,j entry equal to $K(x_i, y_j)$. We will use the notation

$$\kappa(x_1, \ldots, x_n; y_1, \ldots, y_n) = \det(L). \tag{4.3}$$

Lemma 4.1. *Let $(a_1, \ldots, a_{n+1}; b_1, \ldots, b_{n+1})$ be a circular pair. Then*

$$\begin{aligned}\kappa(a_1, \ldots, a_{n+1}; b_1, \ldots, b_{n+1})\kappa(a_1, \ldots, a_{n-1}; b_3, \ldots, b_{n+1}) = \\ \kappa(a_1, \ldots, a_n; b_1, b_3, \ldots, b_{n+1})\kappa(a_1, \ldots, a_{n-1}, a_{n+1}; b_2, \ldots, b_{n+1}) \\ - \kappa(a_1, \ldots, a_n; b_2, \ldots, b_{n+1})\kappa(a_1, \ldots, a_{n-1}, a_{n+1}; b_1, b_3, \ldots, b_{n+1})\end{aligned} \tag{4.4}$$

Assume that

$$\kappa(x_1, \ldots, x_n; y_1, \ldots, y_n) = 0 \tag{4.5}$$

for some circular pair. First we claim that there is no sequence of points z_j such that $x_n < z_j < y_1$, $\lim_{j\to\infty} z_j = x_n$, and $\lim_{j\to\infty} \kappa(x_1, \ldots, x_n; z_j, y_2, \ldots, y_n) = 0$. For this would imply that there are constants c_k (independent of j) so that

$$K(x_n, z_j) = \sum_{k<n} c_k K(x_k, z_j), \tag{4.6}$$

and hence

$$\lim_{j\to\infty} K(x_n, z_j) = \sum_{k<n} c_k K(x_k, x_n), \tag{4.7}$$

contradicting (1.4). Thus there is a number η_1 with $x_n < \eta_1 < y_1$ such that

$$\kappa(x_1, \ldots, x_n; \eta_1, y_2, \ldots, y_n) = 0 \text{ and} \tag{4.8}$$

$$\kappa(x_1, \ldots, x_n; \eta, y_2, \ldots, y_n) \neq 0, \text{ for } x_n < \eta < \eta_1. \tag{4.9}$$

Let x be such that $x_n < x < \eta_1$. Then there is an η such that $x < \eta < \eta_1$ and hence $(x_1, \ldots, x_n, x; \eta, \eta_1, y_2, \ldots, y_n)$ is a circular pair. By (4.4), (4.5), and (3.1)

$$\begin{aligned}0 \geq \kappa(x_1, \ldots, x_n, x; \eta, \eta_1, y_2, \ldots, y_n)\kappa(x_1, \ldots, x_{n-1}; y_2, \ldots, y_n) = \\ \kappa(x_1, \ldots, x_n; \eta, y_2, \ldots, y_n)\kappa(x_1, \ldots, x_{n-1}, x; \eta_1, y_2, \ldots, y_n) \\ - \kappa(x_1, \ldots, x_n; \eta_1, y_2, \ldots, y_n)\kappa(x_1, \ldots, x_{n-1}, x; \eta, y_2, \ldots, y_n) \\ = \kappa(x_1, \ldots, x_n; \eta, y_2, \ldots, y_n)\kappa(x_1, \ldots, x_{n-1}, x; \eta_1, y_2, \ldots, y_n) \geq 0.\end{aligned} \tag{4.10}$$

Using this and (4.9) we see that

$$\kappa(x_1, \ldots, x_{n-1}, x; \eta_1, y_2, \ldots, y_n) = 0, \text{ for } x_n < x < \eta_1. \tag{4.11}$$

As above, this contradicts (1.4) and proves the theorem.

5. The Hopf Lemma

We now show how the fact that $K(x, y) < 0$ for $x \neq y$ implies the Hopf lemma (reference) for the conductivity equation.

Theorem 5.1. *Let u be a non constant solution of $\nabla(\gamma\nabla u) = 0$, and let $p \in \partial\Omega$ be a point where u assumes a minimum. Then*

$$\frac{\partial u}{\partial n}(p) < 0 \tag{5.1}$$

Proof. We may assume that $u(p) = 0$. Let $f = u|_{\partial\Omega}$. Since u is not constant, $\mathrm{supp}(f)$ is not empty. Thus there is an interval D around p in $\partial\Omega$ such that $\mathrm{supp}(f) - D$ is not empty. Let ψ be a smooth function on $\partial\Omega$ such that $\psi = 1$ on $\mathrm{supp}(f) - D$, $\psi = 0$ on an interval around p, and $0 \leq \psi \leq 1$. Let $g = \psi f$ and let v be the solution of $\nabla(\gamma\nabla v) = 0$ with $v|_{\partial\Omega} = g$. Since $f \geq g$ it follows that $u \geq v$. It is also true that $g \geq 0$. Since $p \notin \mathrm{supp}(g)$ and $K(p, y) < 0$,

$$0 > \int_{\partial\Omega} K(p, y)g(y)dy = \gamma(p)\frac{\partial v}{\partial n}(p) \geq \gamma(p)\frac{\partial u}{\partial n}(p), \tag{5.2}$$

which proves the theorem. □

6. The Variation Diminishing Property

We will use the following notation. Let $M(x, y)$ be a continuous function on $[c, d]\times[a, b]$. Let $c \leq x_1 < x_2 < \cdots < x_n \leq d$, $a \leq y_1 < y_2 < \cdots < y_n \leq b$. Let T be the $n \times n$ matrix with i, j entry equal to $M(x_i, y_j)$. Let

$$\mu(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n) = \det(T).$$

The following lemma from [5] is sometimes paraphrased by saying that the kernel M has the *variation diminishing property.* It will be used to show that the inequalities (1.5) imply the alternating property.

Lemma 6.1. *Let f be a continuous, not identically 0, function defined on the interval $[a, b]$, such that f changes its sign on this interval no more than $n - 1$ times. Let $M(x, y)$, $x, y \in [c, d] \times [a, b]$, be a continuous kernel with the property that*

$$\mu(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n) > 0, \tag{6.1}$$

whenever $c \leq x_1 < x_2 < \cdots < x_n \leq d$, $a \leq y_1 < y_2 < \cdots < y_n \leq b$. Then the function

$$g(x) = \int_a^b M(x, y)f(y)dy$$

vanishes in $[c, d]$ no more than $n - 1$ times.

By saying that function f changes its sign k times on the interval $[a, b]$ we mean that there are $k + 1$ points $x_1 < x_2 < \cdots < x_{k+1}$ in $[a, b]$ such that for $i = 1, 2, \ldots, k$

$$f(x_i)f(x_{i+1}) < 0. \tag{6.2}$$

Proof. By hypothesis there are points $a = s_0 < s_1 < s_2 < \cdots < s_{n-1} < s_n = b$ such that in each interval (s_{i-1}, s_i), $i = 1, 2, \ldots, n$ function f does not change its sign and is not identically 0. For $i = 1, 2, \ldots, n$ let

$$g_i(x) = \int_{s_{i-1}}^{s_i} M(x, y) f(y) dy. \tag{6.3}$$

Then

$$g(x) = \sum_{i=1}^{n} g_i(x). \tag{6.4}$$

For any $c \leq x_1 < x_2 < \cdots < x_n \leq d$ the determinant
(6.5)

$$\det(\{g_i(x_j\}) = \int_{s_{n-1}}^{s_n} \cdots \int_{s_0}^{s_1} \mu(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n) f(y_1) \ldots f(y_n) dy_1 \ldots dy_n$$

is not 0 since the integrand is not identically zero and has constant sign. This shows that there is no non-trivial linear combination of g_i's vanishing at n points and hence that $g(x) = \sum_{i=1}^{n} g_i(x)$ cannot vanish at n points. □

We note that this proof only used the fact that $\mu(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n)$ has constant sign. We need one more lemma before coming to the proof of the alternating principal.

Let $K(x, y)$ be a kernel on $\partial\Omega \times \partial\Omega$. We assume that $K(x, y)$ is continuous when $x \neq y$, but we don't assume anything about K on the diagonal of $\partial\Omega \times \partial\Omega$. Let $\kappa(x_1, \ldots, x_n; y_1, \ldots, y_n)$ be defined as in section 4.

Lemma 6.2. *Suppose that $\kappa(x_1, \ldots, x_n; y_1, \ldots, y_n)$ is never zero and has constant sign for all circular n-pairs $(x_1, \ldots, x_n; y_1, \ldots, y_n)$. Let $\partial\Omega = I \cup J$ where I and J are disjoint connected arcs. Let f be a continuous function on $\partial\Omega$ with supp$(f) \subset J$. Let*

$$g(x) = \int_{\partial\Omega} K(x, y) f(y) dy. \tag{6.6}$$

Then if there is a sequence of $n + 1$ points in I in circular order at which g alternates in sign, then there is a sequence of at least $n + 1$ points in J in circular order at which f alternates in sign.

Proof. If there is no sequence of $n + 1$ points of J at which f alternates in sign, then f can change its sign no more than $n - 1$ times in J. By Lemma 6.1, g can vanish no more than $n - 1$ times in I. But we are assuming that g has $n + 1$ alternations of sign in I and hence at least n zeros in I. This contradiction proves the lemma. □

We now state and prove the theorem.

Theorem 6.1. *Using the notation of Lemma 6.2, suppose that*

$$(-1)^{\frac{n(n+1)}{2}} \kappa(x_1, \ldots, x_n; y_1, \ldots, y_n) > 0 \tag{6.7}$$

for all $n > 0$ and all circular n-pairs $(x_1, \ldots, x_n; y_1, \ldots, y_n)$. Let f be a continuous function on $\partial\Omega$ with supp$(f) \subset J$. Let

$$g(x) = \int_{\partial\Omega} K(x,y)f(y)dy. \tag{6.8}$$

Suppose there is a sequence of points $\{p_1, \ldots, p_n\} \subset I$ in circular order such that

$$(-1)^{i+1} g(p_i) > 0 \tag{6.9}$$

Then there is a sequence of points $\{q_1, \ldots, q_n\} \subset J$ in circular order such that

$$(-1)^n g(p_i) f(q_i) > 0. \tag{6.10}$$

Proof. By Lemma 6.2 there is a sequence of points in J at which f alternates in sign. If there is no sequence with the desired alteration property then J is a disjoint union of subintervals J_i, in circular order, such that

(1) f is not identically 0 on J_i, $i = 1, \ldots, n$,
(2) f does not change its sign on J_i, $i = 1, \ldots, n$
(3) for some $z_i \in J_i$,

$$(-1)^{n+i} f(z_i) > 0. \tag{6.11}$$

We use the idea of Lemma 6.1. For $i = 1, 2, \ldots, n$ let

$$g_i(x) = \int_{J_i} K(x,y)f(y)dy. \tag{6.12}$$

Then

$$g(x) = \sum_{i=1}^{n} g_i(x). \tag{6.13}$$

Let

$$G = \begin{bmatrix} g_1(x_1) & g_2(x_1) & \ldots & g_n(x_1) \\ g_1(x_2) & \ldots & & g_n(x_2) \\ \vdots & & & \vdots \\ g_1(x_n) & \ldots & & g_n(x_n) \end{bmatrix}. \tag{6.14}$$

Let u be the n-vector with $u_i = 1$, $i = 1, \ldots, n$. Then

$$Gu = \begin{bmatrix} g(x_1) \\ g(x_2) \\ \vdots \\ g(x_n) \end{bmatrix}. \tag{6.15}$$

Using (6.11) we will show that the signs of u are all negative. This contradiction will prove the theorem. We need to compute the signs of the entries of G^{-1}. Rather than get lost in a cloud of indices, we will give the proof in the case that $n = 3$ and leave the general proof to the reader. In this case the assumption (6.11) implies that $f(y) \geq 0$ in J_1, $f(y) \leq 0$ in J_2, and

$f(y) \geq 0$ in J_3. As in section 3 we compute the signs of the cofactors of G. First we have

(6.16)
$$\det(G) = \int_{J_1}\int_{J_2}\int_{J_3} \kappa(x_1,x_2,x_3;y_1,y_2,y_3)f(y_1)f(y_2)f(y_3)dy_1dy_2dy_3 < 0.$$

We find that

$$\begin{vmatrix} g_2(x_2) & g_3(x_2) \\ g_2(x_3) & g_3(x_3) \end{vmatrix} = \int_{J_2}\int_{J_3} \kappa(x_2,x_3;y_2,y_3)f(y_2)f(y_3)dy_2dy_3 > 0. \tag{6.17}$$

Hence $(G^{-1})_{11} < 0$. Next we compute that

(6.18)
$$(-1)^{1+2}\begin{vmatrix} g_1(x_2) & g_3(x_2) \\ g_1(x_3) & g_3(x_3) \end{vmatrix} = \int_{J_1}\int_{J_3} \kappa(x_2,x_3;y_1,y_3)f(y_1)f(y_3)dy_1dy_3 > 0,$$

and thus $(G^{-1})_{21} < 0$. Continuing the calculation we find that the signs of G^{-1} are as follows

$$G^{-1} = \begin{bmatrix} - & + & - \\ - & + & - \\ - & + & - \end{bmatrix}. \tag{6.19}$$

This yields the contradiction

$$\begin{bmatrix} - & + & - \\ - & + & - \\ - & + & - \end{bmatrix}\begin{bmatrix} + \\ - \\ + \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \tag{6.20}$$

□

7. Remarks and Conjectures

We have not tried to state the most general hypotheses under which our results are valid, but have stated them in such a way that the essential ideas of the proofs are clear. We can also prove determinant inequalities for for certain "blocks" in Dirichlet-to-Neumann kernels for multiply connected plane domains. We can differentiate our inequalities to get a set of inequalities involving determinants of derivatives of the Dirichlet-to-Neumann kernel. These inequalities are equivalent to the set of inequalities (1.5). In our arguments we seem to need to assume that γ is in $C^2(\Omega)$, however it is possible that weaker assumptions would suffice.

We would like to single out the following conjecture on characterizing the kernel of a Dirichlet-to-Neumann map.

Conjecture 7.1. *Let Ω be a relatively compact, simply connected region in the plane with C^2 boundary. Let $K(x,y) = \frac{k(x,y)}{|x-y|^2}$, where $(x,y) \in \partial\Omega \times \partial\Omega - \Delta$, k is continuous on $\partial\Omega \times \partial\Omega$, $k(x,x) \neq 0$, and K satisfies (1.5). Then there is a distribution $D(x,y)$ on $\partial\Omega \times \partial\Omega$, supported on the diagonal, Δ, and a regularization of K as a distribution on $\partial\Omega \times \partial\Omega$, so that $L = K + D$*

is the kernel of the Dirichlet-to-Neumann map for some conductivity, γ, on Ω. The distribution D is determined by the property that

$$\int_{\partial\Omega} L(x,y)dy = 0. \tag{7.1}$$

Equation (7.1) is analogous to the fact the the Dirichlet-to-Neumann matrix for an electrical network has row sums equal to zero. This implies that the diagonal is determined by the off-diagonal terms. This is true as well in the continuous case.

We have discussed these results with many people and their suggestions and advice have been of great benefit. Among these people are Ed Curtis, John Sylvester, and Gunther Uhlmann. The importance of the alternating property was recognized some time ago by Ed Curtis.

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