Examples on Typesetting Commutative Diagrams
Using \texttt{Xy-pic}

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\textit{Edition 1}

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This printout provides examples on typesetting commutative diagrams using \texttt{Xy-pic}'s \texttt{\textbackslash xymatrix\{\ldots\}} command which view commutative diagrams as “matrix-like diagrams”.

The printout is an attempt to introduce the complete newcomer to \texttt{Xy-pic}.

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Chapter 1

On Typesetting Commutative Diagrams

1.1 Introduction

In category theory, “commutative diagrams” are the categorists ways to illustrate equations and universal properties. Here is an example:

Let $\mathcal{E}$ be a topos, and let $\Omega$ be its subobject-classifier. A Lawvere-Tierney topology on $\mathcal{E}$ is a map $j : \Omega \rightarrow \Omega$ in $\mathcal{E}$ such that the following three diagrams commute.

$$
\begin{array}{c}
1 \xrightarrow{\text{true}} \Omega \\
\downarrow_{\text{true}} \quad \downarrow_{j} \\
\Omega & \quad \Omega
\end{array}
\quad
\begin{array}{c}
\Omega \xrightarrow{j} \Omega \\
\downarrow_{j} \\
\Omega
\end{array}
\quad
\begin{array}{c}
\Omega \times \Omega \xrightarrow{\wedge} \Omega \\
\downarrow_{j \times j} \\
\Omega \times \Omega \xrightarrow{j}
\end{array}
$$

\[(j \circ \text{true} = \text{true}) \quad (j \circ j = j) \quad (j \circ \wedge = \wedge \circ (j \times j))\]

There are a few programs to draw commutative diagrams. Two of the best are Xy-pic and PSTricks.

Xy-pic package is © by its authors as free software for typesetting graphs and diagrams such as arrows, curves, frames, directed graphs, paths, polygons, knots, and commutative diagrams and 2-cell structures as in 2-categories. It was written by: Kristoffer Rose: <Kristoffer.Rose@ENS-Lyon.FR>, and Ross Moore: <ross@ics.mq.edu.au>.

For complete information on Xy-pic diagrams in general, please refer to the Xy-guide and Xy-reference manual, which can be downloaded and printed out, from one of the following sites:

<http://www.ens-lyon.fr/~krisrose/Xy-pic.html>
Overall Structure of a Document

\textit{Xy-pic} works with most formats (including \LaTeX, \texttt{AMSTeX}, \texttt{AMS-TeX}, and plain \TeX). It must be loaded into your format working memory.

Here is a sample of the overall structure of a \LaTeX Document:

\begin{verbatim}
\documentclass[12pt]{report}
\usepackage{geometry,amsthm,graphics,amsymb,amsmath,enumerate,latexsym, tabularx,shapepar}
\usepackage[all,2cell,dvips]{xy} \UseAllTwocells \SilentMatrices
\begin{document}
  ...
  ...
  ...
  ...
  ...
\end{document}
\end{verbatim}

The control sequences
\begin{verbatim}
\usepackage[all,2cell,dvips]{xy} \UseAllTwocells \SilentMatrices
\end{verbatim}
are the ones I used to produce this document.
1.2 The Basic Construction

The \texttt{\textbackslash xymatrix{...}} command views a commutative diagram as a "matrix-like diagram"; a matrix that has "vertices" or "entries". To get an idea, suppose we want to typeset the diagram

$$
\begin{array}{c}
A \\
\downarrow \\
C \\
\end{array}
\begin{array}{c}
B \\
\downarrow \\
D \\
\end{array}
$$

First, view the vertices of the diagram

$$
\begin{array}{cc}
A & B \\
C & D \\
\end{array}
$$

as entries of a matrix. Any two entries can serve as a source and target for an arrow. An arrow is set by typing the command that produces that arrow \texttt{right after} the source entry where the arrow starts. Such command takes the form \texttt{\textbackslash ar[direction]} where the variable "direction" takes one value as illustrated by the diagram:

$$
\begin{array}{c}
l u \\
\downarrow \\
l d \\
\end{array}
\begin{array}{c}
u \\
\downarrow \\
d \\
\end{array}
\begin{array}{c}
ru \\
\downarrow \\
rd \\
\end{array}
\begin{array}{c}
l \\
\downarrow \\
r \\
\end{array}
\begin{array}{c}
l u \\
\downarrow \\
l d \\
\end{array}
$$

For example, \texttt{\textbackslash ar[r]} produces a right (east) arrow from its source, and \texttt{\textbackslash ar[rd]} produces a right-down (southeast) arrow from its source. In addition, other combinations of directions are possible. For example, \texttt{\textbackslash ar[rru]} will produce a right-right-up (east-northeast) arrow from its source, and \texttt{\textbackslash ar[rdd]} will produce a right-down-down (south-southeast) arrow from its source, and so on.

\textit{Note: It has the same affect to use [ru] or [ur]. Similarly, all other combination of directions.}

Now let us apply what we have said. Our commutative diagram

$$
\begin{array}{c}
A \\
\downarrow \\
C \\
\end{array}
\begin{array}{c}
B \\
\downarrow \\
D \\
\end{array}
$$

can be typed as

\texttt{\textbackslash xymatrix{ A \textbackslash ar[r] \textbackslash ar[d] & B \textbackslash ar[d] \textbackslash ar[ld] \\
C \textbackslash ar[r] & D}}
Here is a second example: the diagram

\[
\begin{array}{cc}
A & B \\
C & D \\
E & F
\end{array}
\]

can be produced by

\texttt{\textbackslash xymatrix{ A \ & B \ \textbackslash ar[1dd]\textbackslash\\
C \ & D\\
E \ & F}}

It's all as simple as that, more or less. Notice how the special & character separates the entries in any one row, and the \textbackslash separates different rows. To center the diagram by itself in a page you may want to enclose it by \texttt{\[ \ldots \]} or \texttt{\$\$ \ldots \$\$}.

If you use \texttt{\textbackslash ar[direction]} where is there is no target, that is, if you point an arrow outside the \texttt{\textbackslash xymatrix{ \ldots \} grid then E\TeX \ will respond by giving an error message to the terminal screen.

In some diagrams, there is no vertex in an entry. Empty vertices may be omitted, but we still need to use the & to separate the columns in order for the arrows be pointing somewhere. In general, empty vertices at the end of rows may be omitted. Here is an example:

\[
\begin{array}{c}
1 \\
\end{array}
\rightarrow \Omega
\]

\texttt{\textbackslash xymatrix{ 1 \ \textbackslash ar[r] \ \textbackslash ar[rd] \ \textbackslash Omega \ \textbackslash ar[d]\textbackslash\\
& \}}

You may prefer to use \texttt{\{\}} or \texttt{\textbackslash} to replace an empty vertex.
1.3 Arrows’ Labels

Arrows can be labeled. The position of labels is specified by \( ^\), \( _\), for the position, and \( |\) for breaking an arrow with a label. The dash \( -\) used, if needed, to center labels. Here are examples. (Notice the reversed meaning of \( ^\) and \( _\) when arrows are reversed from right to left and from down to up.)

- for right arrows, \( \ar[r] f \) produces \( \rightarrow \)
- for right arrows, \( \ar[r] ^f g \) produces \( \frac{f}{g} \)
- for left arrows, \( \ar[l] ^f g \) produces \( \frac{g}{f} \)
- for right down arrows, \( \ar[rd] f \) produces \( \downarrow \)
- for left up arrows, \( \ar[lu] f \) produces \( \uparrow \)

Similarly, other-direction arrows.

Vertices’ Long Names

\( \text{Xy-pic} \) essentially lays down the vertices first, and then superimposes the arrows upon them. The size of the individual vertices and the sizes of the column and row gaps are taken into account when the vertices are first printed, but their position is not influenced by the arrows, and, in particular, not by any labels on the arrows. This means that a long vertice can causes the arrow label to be positioned not in the middle of the arrow, here is an example:

\[
\text{\texttt{xymatrix}}\{ M \ar[r]^{\theta} \theta_{\text{\&}} S \otimes M \otimes S \}
\]

\[
M \rightarrow \theta S \otimes M \otimes S
\]

This is because \( \text{Xy-pic} \) places the label between the centers of the source and target objects. To correct this, so that the label can centered in the arrow, add a dash \( -\) right before the label, as follows:

\[
\text{\texttt{xymatrix}}\{ M \ar[r]^{-\theta} \theta_{\text{\&}} S \otimes M \otimes S \}
\]

\[
M \rightarrow \theta S \otimes M \otimes S
\]
Arrows’ Long Labels

Also, a long arrow’s label causes the diagram to be cramped. For example,

\[
\text{xymatrix}{(d_0)_! (d_0)_! d^* 2d^* 1}
\ar[r]^-{\tilde{\text{d}}_1= (d_0)_! \epsilon^{-1} (d_1)_! d^*}
\ar[r]|-{} (d_0)_! d^*_1
\]

This problem can be solved by adding extra columns of ‘empty’ vertices in the diagram. For example, adding two columns, and using the dash - produce

\[
\text{xymatrix}{(d_0)_! (d_0)_! d^* 2d^* 1}
\ar[rrr]^-{\tilde{\text{d}}_1= (d_0)_! \epsilon^{-1} (d_1)_! d^*}
\ar[r]|-{} (d_0)_! d^*_1
\]

Another solution is by increasing the column gap between the two vertices. For example, adding the option @C=2.5pc\(^1\), just after the command \text{xymatrix}:

\[
\text{xymatrix@C=2.5pc}{(d_0)_! (d_0)_! d^* 2d^* 1}
\ar[rrr]^-{\tilde{\text{d}}_1= (d_0)_! \epsilon^{-1} (d_1)_! d^*}
\ar[r]|-{} (d_0)_! d^*_1
\]

\(^1\) See the section about measurements in \text{T}_{\text{E}}\text{X}.
Moving Labels

Labels can be slid along an arrow, if needed, by adding the option \texttt{number} just before the label. Here are examples:

\[
\begin{array}{|c|}
\hline
\text{xymatrix}{M \ar[r]^\theta & S} \\
\text{xymatrix}{M \ar[r]^\theta & S} \\
\text{xymatrix}{M \ar[r]^\theta & S} \\
\text{xymatrix}{M \ar[r]^\theta & S} \\
\text{xymatrix}{M \ar[r]^\theta & S} \\
\text{xymatrix}{M \ar[r]^\theta & S} \\
\end{array}
\]

In a particular situation, trial and error may be helpful in choosing a suitable position for a label.
Left and Right Shifts of vertices

The position of a particular entry in a commutative diagram can be modified, if needed, by entering **[1] or **[r] just before the entry. The first produces a left shift in the object by almost one-half of its width, and the second produces a right shift in the object by almost one-half of its width. Compare the following two diagrams:

```
\[
\xydiagram{
A \oplus B \oplus C \ar[r] \ar[d] & \ar[d] & A \oplus B \oplus C
\}
\]

\[
\xydiagram{
A \oplus B \oplus C \ar[r] \ar[d] & B \\
A \ar[r] & A \oplus B \oplus C
\}
\]
```

Notice the affect on the horizontal arrows.
1.4 Length Measurements in \TeX

To change size of diagrams or text in \TeX, and its packages, we may utilize the units by which \TeX can measure lengths. Here is a table to give an idea of the comparative sizes:

<table>
<thead>
<tr>
<th>Unit</th>
<th>Illustration</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 inch (in)</td>
<td>-</td>
<td>1 point is about the size of a period. 1 inch=72.27 points.</td>
</tr>
<tr>
<td>1 centimeter (cm)</td>
<td>-</td>
<td>1 pica=12 points</td>
</tr>
<tr>
<td>1 millimeter (mm)</td>
<td>-</td>
<td>1 em is a little smaller than the width of a capital \text{M}.</td>
</tr>
<tr>
<td>1 point (pt)</td>
<td>.</td>
<td>1 ex is about the height of a small \text{x}</td>
</tr>
<tr>
<td>1 pica (pc)</td>
<td>-</td>
<td>1 em is a little smaller than the width of a capital \text{M}.</td>
</tr>
</tbody>
</table>

1.5 Spacing Between Rows and Columns of a Commutative Diagram

A code such as
\begin{verbatim}
\xymatrix{
  A \ar[r]^\theta \approx \ar[d]_m & B \\
  C \ar[r] & D
}\end{verbatim}

will produce a commutative diagram whose size is determined by default.

\begin{equation}
\begin{array}{c}
  A \xymatrix{ \ar[r]^\theta \approx \ar[d]_m & B \\
  \ar[d]_m & \ar[d]^n & \\
  C & \ar[r] & D
  \end{array}
\end{equation}

Its size can be increased by adding the option \text{\textbackslash p}{=2.5pc}, for example, just after the command \textbackslash \xymatrix, as follows:
\text{xymatrix}{
A \ar[r] `\theta \ar[d] \approx \ar[d] & B
m \ar[d] \approx n
C \ar[r] & D
}

As you see, $2.5\text{pc}$ has a uniform affect on length and width. We may choose to adjust spacing between rows and between columns for certain special effects. This can be done by adding the options $\text{\textbackslash R=some_length}$ and $\text{\textbackslash C=some_length}$ just after the command $\text{xymatrix}$. Here is an example:

\text{xymatrix}{
\text{\textbackslash R=.4in} \text{\textbackslash C=1.5in}
A \ar[r] `\theta \ar[d] \approx \ar[d] & B
m \ar[d] \approx n
C \ar[r] & D
}

Here is another example: a diagram like

\[
\text{xymatrix}{
H_* \ar[r] B \otimes B \ar[r] \otimes \ar[d] & H_* \ar[d] \\
H_* \ar[r] (B \otimes B) \ar[r] \otimes \ar[d] & H_* (B \otimes B) \ar[d]
}
\]

$H_* (B \otimes B)$

$\text{\textbackslash R=r}$

$H_* (B \otimes B) \otimes H_* (B \otimes B)$

looks better if the spacing between rows is increased, and between columns is decreased, as follows:
1.6 Parallel Arrows

To get a pair of parallel arrows between the same two vertices, move the arrow parallel
to itself a small number of distance units. Here is an example:

\begin{xy}
\xymatrix{ X_1 \ar@<+.7ex>[r] \ar@<-.7ex>[r] & X_0 }
\end{xy}

where \texttt{+.7ex} caused the arrow to be moved parallel to itself up by .7 ex, and
\texttt{-.7ex} caused the arrow to be moved parallel to itself down by .7 ex. Similarly,
other-direction arrows.

Here is another example in which we need to modify distances in the diagram to
improve it. In the following diagram, the arrow and the target object are ‘lower’ than
the source object:

\begin{xy}
\xymatrix{ \times_{x \in U_{iy}} A_x \ar[r] & A_y }
\end{xy}

The diagram can be improved by moving the arrow parallel to itself by 0.8 ex as
follows:

\begin{xy}
\xymatrix{ \times_{x \in U_{iy}} A_x \ar[r] & A_y }
\end{xy}
1.7 Arrow Styles

Any arrow has a tail, a shaft, and a head. In the default case

\[ A \xrightarrow{f} B \]

the tail is empty, the shaft is a solid line, and the head is an arrow head as shown. \texttt{Xy-pic} comes with a large number of arrow styles, where each of the three parts (tail, shaft, head) of an arrow can be changed. The following tables contain a list that I kept for myself as a reference about arrow styles.

<table>
<thead>
<tr>
<th>Objective and Input</th>
<th>Output (what you get)</th>
</tr>
</thead>
<tbody>
<tr>
<td>The default case arrow: ( \xymatrix{ A \ar[r] &amp; B } )</td>
<td>( A \xrightarrow{f} B )</td>
</tr>
<tr>
<td>Inclusion arrow: ( \xymatrix{ A \ar@{&lt;-&gt;}[r] &amp; B } )</td>
<td>( A \xrightarrow{f} B )</td>
</tr>
<tr>
<td>Epimorphism arrow: ( \xymatrix{ A \ar[r] &amp; B } )</td>
<td>( A \xrightarrow{f} B )</td>
</tr>
<tr>
<td>Monomorphism arrow: ( \xymatrix{ A \ar@{-&gt;}[r] &amp; B } )</td>
<td>( A \xrightarrow{f} B )</td>
</tr>
<tr>
<td>A function’s action on an element: ( \xymatrix{ A \ar[r] &amp; B } )</td>
<td>( A \xrightarrow{f} B )</td>
</tr>
<tr>
<td>Dotted arrow: ( \xymatrix{ A \ar@{-&gt;}[r] &amp; B } )</td>
<td>( A \xrightarrow{f} B )</td>
</tr>
<tr>
<td>Dashed arrow: ( \xymatrix{ A \ar[r] &amp; B } )</td>
<td>( A \xrightarrow{f} B )</td>
</tr>
<tr>
<td>Squiggle arrow: ( \xymatrix{ A \ar@{-}[r] &amp; B } )</td>
<td>( A \xrightarrow{f} B )</td>
</tr>
</tbody>
</table>

12
| Long equal sign:   | \( \begin{xy} \
A \ar@{=}@[r] \& \text{B} \end{xy} \) | \( A \LongEqual B \) |
|--------------------|--------------------------------------------------|------------------|
| Double dotted arrow: | \( \begin{xy} \
A \ar@{::}@[r] \& \text{B} \end{xy} \) | \( A \DottedDoubleEqual B \) |
| Double arrow:       | \( \begin{xy} \
A \ar@{>}@[r] \& \text{B} \end{xy} \) | \( A \DoubleArrow B \) |
| Double arrow:       | \( \begin{xy} \
A \ar@{>}@[r] \& \text{B} \end{xy} \) | \( A \DoubleArrow B \) |
| Double arrow:       | \( \begin{xy} \
A \ar@{<>}@[r] \& \text{B} \end{xy} \) | \( A \DoubleArrow B \) |
| Triple arrow:       | \( \begin{xy} \
A \ar@{3}@[r] \& \text{B} \end{xy} \) | \( A \TripleArrow B \) |
Arrows’ Heads

It is possible to select arrows’ heads (tips) from certain fonts. Simply add \SelectTips{tip’s family}{tip’s size} just before the \xymatrix command. Here is a table showing the possible tip’s families and sizes:

<table>
<thead>
<tr>
<th>Family</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>xy fonts</td>
<td>→</td>
<td>→</td>
<td>→</td>
</tr>
<tr>
<td>computer modern fonts</td>
<td>→</td>
<td>→</td>
<td>→</td>
</tr>
<tr>
<td>Euler math fonts</td>
<td>→</td>
<td>→</td>
<td>→</td>
</tr>
</tbody>
</table>

For example, the arrow in

\xymatrix{A
\ar[r] & B}

\( A \rightarrow B \)

has a head that comes by default, but using Euler math font size 12, we get:

\SelectTips{eu}{12}\xymatrix{A
\ar[r] & B}

\( A \rightarrow B \)

Once a selection is made, it has an affect on all of the commutative diagrams that follow. To stop the affect, use the command \NoTips at the place, in your document, where you want to deactivate the selection and go back using the default setting.

Here is a second example on selecting heads:

\xymatrix{01
\ar[r] |->{\SelectTips{cm}{}}\object{>>}}
|->{\SelectTips{eu}{}\object{>>}}
& B }

\( A \rightarrow B \)

The type of arrow’s head can be selected for an entire \LaTeX{} document by the declaration

\SelectTips{tip’s family}{tip’s size} in the document preamble.
1.8 Cubes and More!

The cube offers a good example of how to typeset a commutative diagram. The following cube has Euler-math-font-size-12 arrow's heads:

\[
\begin{xy}
0;/r1pc/: A -> B ; A' -> B' ; C -> D ; C' -> D' \end{xy}
\]

It is produced by the code:

\SelectTips{eu}{12}
\begin{verbatim}
\[
\begin{xy}
0;/r1pc/: A \ar[d] \ar[rd] \ar[rr] \& B \ar[d] \ar[rd]
& A' \ar[d] \ar[rr] \& B' \ar[d]
C \ar[r] \ar[rr] \& D \ar[d]
& C' \ar[r] \& D'
\end{xy}
\]
\end{verbatim}

Notice the use of \texttt{'}\texttt{[d]} and \texttt{'}\texttt{[r]} to make holes in the arrows.

The following prism has xy-font-size-12 arrow's heads:

\[
\begin{xy}
0;/r1pc/: A -> C ; B -> C' ; A' -> B' ; A' -> B' ; A' -> B' \end{xy}
\]

It is produced by the code:

\SelectTips{xy}{12}
\begin{verbatim}
\[
\begin{xy}
0;/r1pc/: A \ar[rr] \ar[rrd] \ar[dd] \& C \ar[dd]
& B \ar[ru] \ar[dd]
A' \ar[rr] \ar[rrd] \ar[rd] \& C' \ar[rd]
& B' \ar[ru]
\end{xy}
\]
\end{verbatim}
The following cube has Computer-modern-font-size-12 arrow’s heads:

\[
\begin{array}{c}
\Sigma^L \\
L \\
L_m \\
G
\end{array}
\quad \begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow
\end{array}
\begin{array}{c}
\Sigma^R \\
R \\
R_m \\
H
\end{array}
\quad \begin{array}{c}
\quad \times \\
\end{array}
\begin{array}{c}
\Sigma^G \\
\quad \\
G_m \\
\quad \\
G
\end{array}
\end{array}
\]

It is produced by the code:

\[
\text{\SelectTips{cm}{12}}\\
\xymatrix@!3pc{ & \Sigma^L \ar[r] & \ar[r] & \Sigma^R \\
L & & & L_r \\
L_m & K_{r,m} & R_m \\
G & & & G_r \\
& \quad \times & \quad & \\
& \Sigma^G & & \Sigma^G
\end{array}
\]

Notice the use of \text{'[1] and '}[r] and \text{'[rrr]} to make holes in the arrows.
\begin{xy}
$$\text{``2 cubes'' matrix.}$$
\begin{align*}
\text{Adapted from source by T. Scavo, 1994-7-18.}
\end{align*}
\end{xy}$$

\begin{xy}
$$\text{{\texttt{xy\textbackslash xymatrix{}}}}$$
\begin{align*}
&\& [-1,1] \ar[rr]^G \ar[dll]_{L_2} & & [-1,1] \\
&\& [0,1] \ar[r]^{F_4} & & [0,1] \\
&\& [0,1] \ar[r]^{2C} & & [0,1] \\
&\& [0,1] \ar[rr]^T \ar[u] & & [0,1] \\
\end{align*}
\end{xy}$$
% '4-Simplex Model'
% Adapted from source by John Duskin, May 2001.

\[
\begin{ytableau}
 x_0 & x_4 & x_0 \\
 x_4 & x_1 & x_4 \\
 x_0 & x_4 & x_4 \\
 x_0 & x_4 & x_0 \\
 x_0 & x_4 & x_0 \\
 x_0 & x_4 & x_0 \\
\end{ytableau}
\]
1.9 Application to adjunction

Here is an application of \texttt{xymatrix} to displaying “adjoint arrows.” In the code below, the option \texttt{@[{-}]} is to produce an arrow without a head or tail. The option \texttt{@[{-}\{<1pt>\}]} is to increase the thickness of the horizontal line “arrow” by 1 point; it requires loading the package (dvisps).

\[
\begin{array}{c}
C & \longrightarrow & B^A \\
\hline
C \times A & \longrightarrow & B \\
A & \longrightarrow & B^C \\
B^C & \longrightarrow & A
\end{array}
\]

of A

of A

of A

of $A^{op}$

\textbf{Side Note:} You may color your arrows or lines, if you wish, by using the option \texttt{@[{-}\{color\}]}, where color can be blue, green, red, yellow, etc. Here is an example, but you will not see the affect in this uncolored document:

\[
\begin{array}{c}
C & \longrightarrow & B^A \\
\hline
C \times A & \longrightarrow & B \\
A & \longrightarrow & B^C \\
B^C & \longrightarrow & A
\end{array}
\]

of A

of A

of A

of $A^{op}$

\texttt{\texttt{xymatrix}{ A\ar@[{-}\{\textcolor{green}\{<1pt>\}\}][rrr] \&\& B}}}

\[
\begin{array}{c}
A & \longrightarrow & B
\end{array}
\]
Chapter 2

Examples on Commutative Diagrams

◊

When you have finished studying all the previous examples, then you will be able to analyze the following collection of codes of commutative diagrams. They are presented in the hope that they will be useful. Several of these need to be refined, which is something I hope to do.

◊
**Objective and Input**

The commands to produce the diagrams displayed on the right are as follows, respectively:

\[
\text{\texttt{\textbackslash ymatrix}@1\{U \& S\texttt{ar[r] \texttt{ar[1]} \& V}\}}
\]

\[
\begin{array}{c}
U \quad \quad \quad \quad \quad S \quad \quad V
\end{array}
\]

Note: The \texttt{@1}, in the code above, is a special code that can be used for “one-line” diagrams to improve the placement on the line.

\[
\text{\texttt{\textbackslash ymatrix}\{}
\text{\texttt{\textbackslash mathsf{C}/\{S_0\}} \texttt{\textbackslash ar@<-.7ex>[r]_--\{p_!\}} \texttt{&}
\text{\texttt{\textbackslash mathsf{C}/\texttt{S}\texttt{ar@<-.7ex}[1]_--\{p^*\}}}
\]

\[
\begin{array}{c}
C/S_0 \quad \quad \quad \quad \quad C/S
\end{array}
\]

\[
\text{\texttt{\textbackslash ymatrix}\{}
\text{\texttt{& A \texttt{ar[r]} \_f \texttt{ar[d]} \_i_A \texttt{\textbackslash ar[d]} \_g \_{-1}}} \\
\text{\texttt{& B \texttt{ar[d]} \_f \_{-1}}} \\
\text{\texttt{C \texttt{ar[r]} \_g \& A \}}
\]

\[
\begin{array}{c}
A \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad B
\end{array}
\]

\[
\begin{array}{c}
C \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad A
\end{array}
\]

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\[
\xymatrixcolsep{2.5pc}\xymatrixrowsep{3pc}
\xymatrix{
0 \ar[r] & S \otimes N \ar[r] & S \otimes M \ar[r]_{\epsilon_2} & S \otimes M \otimes S \\
0 \ar[r] & M \ar[r] & M \otimes S \ar[r]_{\epsilon_2} & M \otimes S \otimes S
}
\]

\[
\xymatrix@!=2.5pc{
X_{\{d_1(\theta)\}}
\ar[r]^{\gamma} & X_{d_0(\theta)}
\ar[d]_{\prod_{\theta \in C_1} X_{d_1(\theta)}}
\ar[r]_{\theta} & C_1 \times C_0 \times X \\
C_1 \ar[r]_{d_0} & C_0
}
\]
\begin{align*}
\mathcal{F}_C \xrightarrow{S} \mathcal{F}_C \xrightarrow{T} \mathcal{F}_C \xrightarrow{\tau} \mathcal{F}_C
\end{align*}

\begin{align*}
1_{S_0} \circ T & \xrightarrow{\eta_s} T \circ T \xrightarrow{1_{T} \circ \eta_s} T \circ 1_{S_0} \\
& \cong \\
& \cong
\end{align*}

\begin{align*}
\mathcal{F}(N \otimes S) \otimes S \cong S \otimes (N \otimes S) & \xrightarrow{1 \otimes \eta} S \otimes M \\
& \xrightarrow{\eta \otimes 1} S \otimes g \\
& \xrightarrow{g \otimes S} M \otimes S
\end{align*}
\[
\begin{align*}
\text{g}_3(s \otimes g(1 \otimes m)) &= g_3 g_1 (s \otimes 1 \otimes m) \\
&= g_2 (s \otimes 1 \otimes m) = g(s \otimes m) \otimes 1 = \varepsilon_2 g(s \otimes m).
\end{align*}
\]
\[
\xymatrixrowsep{2pc} \xymatrixcolsep{2pc} \\
\xymatrix{
& S_1 \ar [dr] \ar [dl] \& S_2 \ar [dr] \ar [dl] & \\
U & \& V & \& W}
\]
\qquad \mapsto \quad \qquad \quad \mapsto
\xymatrix@! =1pc{
\& S_1 \times V S_2 \ar [dr] \ar [dl] & \\
S_1 \ar [dl] \ar [dl] \& S_2 \ar [dl] \ar [dl] & \\
U & \& V & \& W}
\]
2.1 Two Cells Diagrams

**Objective and Input**

The commands to produce the diagrams displayed on the right are as follows, respectively:

\[
\xymatrix{ A \rtwo\ell f_{-g}\{\alpha\} \ar[r] & B }
\]

\[
\xymatrix{ A \rtwo\ell<10> f_{-g}\{\alpha\} 
\]

Notice the affect of $<10>$.

\[
\xymatrix{ A \rtwo\ell f_{-g}\{\alpha\} 
\]

Notice the reversed meaning of $\_\_\_\_$ and $\Rightarrow$.

\[
\xymatrix{ A \rtwo\ell f_{-g}\{\text{omit}\} \ar[r] & B }
\]
Objective and Input

The commands to produce the diagrams displayed on the right are as follows, respectively:

\[
\xymatrix{ A & B \ltwocell_f^g{\alpha} \\
A & B \ltwocell_f^g{\alpha} \\
A & B \ruppertwocell^f{\alpha} \rowntwocell_h{\beta} \ar[r]_(.35)g & B }
\]
\xymatrix{ 
& B \\
A \urrtwocell^f_g \quad \alpha & 
\quad \text{quad} \\
\text{xymatrix} \\
& A \ddrrtwocell \\
& & B }$

\xymatrix{ 
& B \\
A \rrrtwocell \quad \zeta_\chi \quad \gamma & 
\quad \text{quad} \\
& C \drruppertwocell \quad \beta & 
& & \text{quad} \\
A \uuruppertwocell \quad \alpha & \text{xlowertwocell[rrrr]} \quad \f & \text{quad} \\
& & B }$
\$\$\texttt{\textbackslash xymatrix@!C=2pc} \\
S_2 \\
\texttt{\textbackslash ruppertwocell<5>\{p\{23\}\{\textbackslash omit}\} \\
\texttt{\textbackslash ar[r]\{p\{13\}\}\texttt{\textbackslash rlowertwocell<5>-\{p\{12\}\}{\textbackslash omit} \& S_1} \\
\texttt{\textbackslash ruppertwocell<5>-\{p\{2\}\}{\textbackslash omit}} \\
\texttt{\textbackslash rlowertwocell<5>-\{p\{1\}\}{\textbackslash omit}\ & S_0\texttt{\textbackslash ar[1]\{e\ \texttt{\textbackslash ar[r]\{p\& S\}}}} \\
\$$

$$ \\
S_2 \xrightarrow{p_{23}} S_1 \xrightarrow{p_{12}} S_0 \xrightarrow{p_1} S \\
$$

$$\$\$\texttt{\textbackslash xymatrix@!C=2pc} \\
\texttt{\textbackslash mathcal\{F\}_{\{S_2\} \&}} \\
\texttt{\textbackslash mathcal\{F\}_{\{S_1\}}\texttt{\textbackslash luppertwocell<5>-\{d^\ast\{2\}\}{\textbackslash omit}}} \\
\texttt{\textbackslash ar[1]\{d^\ast\{1\}\} \texttt{\textbackslash llowertwocell<5>-\{d^\ast\{0\}\}{\textbackslash omit} \texttt{\textbackslash ar[r]\{e^\ast\}}} \& \\
\texttt{\textbackslash mathcal\{F\}_{\{S_0\}}\texttt{\textbackslash luppertwocell<5>-\{d^\ast\{1\}\}{\textbackslash omit}}} \\
\texttt{\textbackslash llowertwocell<5>-\{d^\ast\{0\}\}{\textbackslash omit}} \} \\
\$$

$$ \\
\texttt{\textbackslash F_{S_2} \xleftarrow{d^\ast_2 \{d^\ast_1} \texttt{\textbackslash F_{S_1} \xrightarrow{d^\ast_1 \texttt{\textbackslash F_{S_0}}}} \\
$$

$$\$\$\texttt{\textbackslash xymatrix@!C=2pc} \\
S_3 \texttt{\textbackslash ruppertwocell<5>-\{d_3\}{\textbackslash omit}} \\
\texttt{\textbackslash ar@<1ex>[r]\{d_2\}\texttt{\textbackslash ar@<-1ex>[r]\{d_1\}} \\
\texttt{\textbackslash rlowertwocell<5>-\{d_0\}{\textbackslash omit} \& S_2} \\
\texttt{\textbackslash ruppertwocell<5>-\{d_2\}{\textbackslash omit}} \\
\texttt{\textbackslash ar[r]\{d_1\}\texttt{\textbackslash rlowertwocell<5>-\{d_0\}{\textbackslash omit} \& S_1} \\
\texttt{\textbackslash ruppertwocell<5>-\{d_1\}{\textbackslash omit}} \\
\texttt{\textbackslash rlowertwocell<5>-\{d_0\}{\textbackslash omit} \& S_0\texttt{\textbackslash ar[1]\{e\}}} \\
\$$

$$ \\
S_3 \xrightarrow{d_3} S_2 \xrightarrow{d_2} S_1 \xrightarrow{d_1} S_0 \\
$$
\[\text{Des}(\varepsilon) \rightarrow U \rightarrow \text{Mod-S}\]

\[\text{Mod-R} \quad \varepsilon \rightarrow \varepsilon_1 \rightarrow \text{Mod-S}\]
\[ \text{Diagram} \]

\[ \text{Diagram} \]

\[ \text{Diagram} \]

\[ \text{Diagram} \]

\[ \text{Diagram} \]
\[ \mathcal{F}_Z \xrightarrow{u^*} \mathcal{F}_X \xrightarrow{f_1} \mathcal{F}_Y \]

\[ \mathcal{F}_Z \xrightarrow{v_!} \mathcal{F}_W \xrightarrow{g^*} \mathcal{F}_Y. \]
\[
\begin{array}{ccc}
T_T & T_{\eta T} & T_{\mu T} \\
\downarrow & \downarrow & \downarrow \\
T_T & T_{\eta T} & T_{\mu T} \\
\end{array}
\]

= 

\[
\begin{array}{ccc}
T_T & T_{\eta T} & T_{\mu T} \\
\downarrow & \downarrow & \downarrow \\
T_T & T_{\eta T} & T_{\mu T} \\
\end{array}
\]
$$\xymatrix@!=1pc{ 
&&\text{TB}\ar[dr]^b & \\
A \ar[rr]^\eta_A \ar[d]_l & \ar[r]^\rho & \ar[r]^b & TB \\
& \ar[r]^a & TA & A \ar[r]^f & B \\
& \ar[r]^\eta_B & \ar[r]^\psi & B \\
}$$

\text{=}

$$\xymatrix@!=1pc{ 
&&\text{TA}\ar[dr]^Tf & \\
A \ar[rr]^\eta_A \ar[d]_l & \ar[r]^\rho & \ar[r]^Tf & TB \\
& \ar[r]^a & TA & A \ar[r]^f & B \\
& \ar[r]^\eta_B & \ar[r]^\psi & B \\
& \ar[r]^\rho & TB & B \\
}$$
forall $\quad \forall A \xrightarrow{f} A', \quad the \ following \ equation \ holds$

\[
A \xrightarrow{f} A', \quad \xrightarrow{\alpha} \xrightarrow{\eta_A} TA \xrightarrow{T_f} TA' \quad = \quad TA \xrightarrow{Tf'} A' \xrightarrow{\eta_A} \xrightarrow{\eta_{A'}} TA' \xrightarrow{T'f'}
\]
$$$
\xymatrixrowsep{4pc} \xymatrixcolsep{6pc}
\begin{align*}
d^*_1 X \ar[r]^-a_-\approx & d^n_0 X \\
d^*_1 Y \ar[r]_-{\xi} & d^n_0 Y
\end{align*}
= 
\begin{align*}
d^*_1 X \ar[r]^-a_-\approx & d^n_0 X \\
d^*_1 Y \ar[r]_-{\xi} & d^n_0 Y
\end{align*}$$
\$\$
\texttt{\xymatrixrowsep{6pc}\xymatrixcolsep{6pc}}
\texttt{\xymatrix{}}
\texttt{\{\ar[r] \{d^*_{-2d^*_1}a=d^*_{-1d^*_1}a\} \ar[d] \{d^*_{-3d^*_2}a\}}
\texttt{\ar[dr] \{d^*_{-1d^*_2}a=d^*_{-2d^*_1}a\} & \{\}}
\texttt{\ar@{}[d1] | (.23) \{} \texttt{\begin{big}\Downarrow \ d^*_{-1}a\alpha\end{big}}
\texttt{\{(.75) \{} \texttt{\begin{big}\Downarrow \ d^*_{-3}a\alpha\end{big}} \}}
\texttt{\ar[u] \{d^*_{-1d^*_0}a\}}
\texttt{\}}
\texttt{\}}
\texttt{\}}
\texttt{\\quad = \quad \texttt{\}}
\texttt{\ar[r] \{d^*_{-2d^*_2}a=d^*_{-1d^*_1}a\} \ar[d] \{d^*_{-3d^*_2}a=d^*_{-2d^*_2}a\}}
\texttt{\ar@{}[d1] | (.23) \{} \texttt{\begin{big}\Downarrow \ d^*_{-2}a\alpha\end{big}}
\texttt{\{(.75) \{} \texttt{\begin{big}\Downarrow \ d^*_{-0}a\alpha\end{big}} & \\}}
\texttt{\ar[u] \{d^*_{-0d^*_0}a\}}
\texttt{\}}
\texttt{\}}
\texttt{\}}
\texttt{\}}
\texttt{\}$$

\[
\begin{align*}
\frac{d_5^* d_2 a}{d_3^* d_1 a} &= \frac{d_5^* d_2 a}{d_3^* d_1 a} \\
\frac{d_5^* d_2 a}{d_3^* d_1 a} &= \frac{d_5^* d_2 a}{d_3^* d_1 a}
\end{align*}
\]
2.2 Curved Arrows

Warning: Using curves can be a quite a strain on \( \LaTeX \)'s memory; you should therefore limit the length and number of curves used on a single page. You may use \texttt{\vfill\hfill} at certain points.

\[
\begin{xy}
S_1 @> 2pc { e \circ \ar @/>1pt/[d] \ar @/>1pt/[r] \ar @/>1pt/[rr] \ar @/>1pt/[rrr] } \ar @<2pt/> [d] \ar @<2pt/> [r] \ar @<2pt/> [rr] \ar @<2pt/> [rrr] &
\end{xy}
\]

\[
\begin{xy}
S_1 \ar[r]^{e \circ \ar} & S_0
S_1 \ar[d]_{d_1} \ar[r]_{d_1} & S_1
S_0 \ar[d]_{d_1} \ar[r]_{d_1} & S_0
\end{xy}
\]

\[
\begin{xy}
S_1 \ar[r]^{e \circ \ar} & S_0
S_1 \ar[d]_{d_1} \ar[r]_{d_1} & S_1
S_0 \ar[d]_{d_1} \ar[r]_{d_1} & S_0
\end{xy}
\]

\[
\begin{xy}
S_0 \ar[r]^{\mathrm{Des}}_{d_0} & S
\end{xy}
\]

\[
\begin{xy}
A \ar[r]^{a=\text{"a}} & B \ar[r]^{b=\text{"b}} & C \ar[r]^{\text{"a} ; \text{"b}} &
\end{xy}
\]
\xymatrix@!2pc{
& \{\text{mathcal F}_U\}_{\{S_1 \times V \times S_2\}} \ar[d,r]^{(p_2)_1} \\
& \{\text{mathcal F}_S\}_{\{S_1\}} \ar[r]_{g_1} \ar[d]^{(t_1)_1} & \{\text{mathcal F}_S\}_{\{S_2\}} \ar[d]^{(t'_1)_1} \\
& \{\text{mathcal F}_T\}_{\{T_1\}} \ar[r]_{\eta_1} & \{\text{mathcal F}_W\}_{\{T_1 \times V \times T_2\}} \ar[d]_{(\pi'_2)_1} \\
& \{\text{mathcal F}_T\}_{\{T_2\}} \ar[u]_{(t_2)_1} \ar[r]_{(d'_2)_1} \ar[u]_{(\eta'_2)_1} & \{\text{mathcal F}_V\}_{\{V\}} \ar[u]_{(t'_2)_1} \ar[r]_{\eta} & \{\text{mathcal F}_U\}_{\{S\}} \ar[u]_{(t'_1)_1} \\
& \{\text{mathcal F}_U\}_{\{S\}} \ar[u]_{(s'_2)} & \{\text{mathcal F}_V\}_{\{V\}} \ar[u]_{(s'_1)} & \{\text{mathcal F}_U\}_{\{S\}} \ar[u]_{(s'_2)}}

\text{\footnotesize{44}}
\[ \begin{align*}
\text{d}_0^*X & \cong k_0^*(d_2^*d_1^*X) \\
& \xrightarrow[k_0^*\{d_2^*\theta\}]{} \\
& \cong k_0^*(d_2^*d_0^*X) \cong d_1^*X \\
\text{d}_2^*d_1^*X & \cong d_1^*d_1^*X \\
& \xrightarrow[d_2^*\theta]{} \\
& \cong d_0^*d_1^*X \\
\text{d}_1^*d_0^*X & \cong d_0^*d_0^*X \\
& \xrightarrow[d_1^*\theta]{} \\
& \cong d_0^*d_0^*X \\
\text{k}_0^*(d_0^*d_0^*X) & \cong d_0^*X
\end{align*} \]
Let $\mathcal{N}=\{m \in M \mid g(1 \otimes m) = m \otimes 1\}$, and observe that $\mathcal{N}$ is the kernel of the pair $\left(\varepsilon_2, g \varepsilon_1\right)$. 

\[
\xymatrixrowsep{2pc}
\xymatrix{
0 \ar[r] & \mathcal{N} \ar[r] & M \ar[r]^{\varepsilon_2} & M \otimes S,}
\]

where the pair $\left(\varepsilon_2, g \varepsilon_1\right)$ is as shown:

\[
\xymatrixrowsep{2pc}
\xymatrix{
M \ar@{->}@/_1pc/[rr]^{\varepsilon_2} \ar@{->}@/^1pc/[rr]^{g \varepsilon_1} \ar[r] & S \otimes M \ar[r] & M \otimes S.
}\]
\[\text{\textbackslash ymatrix@1f A \textbackslash ar@<-2pt> 'u[r] ' [r] [r] \textbackslash ar@<-2pt> 'u[r] ' [r] [r] & B }\]

\[A \Rightarrow B\]

\[\text{\textbackslash ymatrix@1f A \textbackslash ar@/-/[r] \textbackslash ar@/-/@<-1ex> [r] & B }\]

\[A \cong B\]

\[\text{\textbackslash ymatrix@C=3pc}{\textbackslash S_1 \textbackslash ar@/-/2pc/ [r]^\text{d_1}\textbackslash ar@/-/1pc/ [r]^\text{d_0}\& \text{\textbackslash S_0}\textbackslash ar@/(dr,ur)[]{id}\textbackslash ar@/-/1pc/ [l]^\text{s_0}}\]

\[S_0 \Rightarrow S_1 \Rightarrow \gamma d\]

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\$\mathrm{xymatrixcolsep{4pc}}\mathrm{xymatrixrowsep{3.9pc}}\$
\xymatrix{
& \{\mathcal{C}_{-1}\} \times \{\mathcal{C}_0\} X \\
\ar[r]^e & \epsilon_{x} \\
\ar@/_/[d1] \{\mathcal{C}_{-1}\} \times \{\mathcal{C}_0\} Y \\
\ar@{->}[d]_{d_{1}} Y \\
\mathcal{C}_{-1} \times \mathcal{C}_{0} \\
\ar[r]_{d_{1}} d_{1} \\
\mathcal{C}_{0} \\
}$

\$\mathrm{xymatrix}{=2.9pc}$
\xymatrix{
X \ar@<2pt>^\eta \ar@{-}[r]_{\epsilon_{x}} \ar@{-}[d]_{x} & \mathcal{C}_{1} \times \mathcal{C}_0 \ar@{-}[r]^{\epsilon_{x}} & X \\
\ar@{-}[r]_{d_{1}} d_{1} & \mathcal{C}_0 \\
}$
2.3 (ROTATION)

Diagrams can be displayed rotated at any direction. The commutative diagram,

```
\$\$
\texttt{\textbackslash xymatrix{S_3 \rightharpoonup \text{ruppertwocell}<5> \{d_3\}\{\text{omit}\}\text{ar}[r]\{d_2\}\text{ar}[l]\{d_1\} \text{rlowertwocell}<5> \{d_0\}\{\text{omit}\}& S_2 \rightharpoonup \text{ruppertwocell}<5> \{d_2\}\{\text{omit}\}\text{ar}[r]\{d_1\}\text{rlowertwocell}<5> \{d_0\}\{\text{omit}\} & S_1 \rightharpoonup \text{ruppertwocell}<5> \{d_1\}\{\text{omit}\} \text{rlowertwocell}<5> \{d_0\}\{\text{omit}\} & S_0 \text{ar}[1]\text{e}\} \$
```

for example, can be rotated to the southwest or southeast directions, by simply adding the options \texttt{\textbackslash xymatrix@dl} and \texttt{\textbackslash xymatrix@dr}, respectively, just after the \texttt{\textbackslash xymatrix} command. Thus,

```
\$\$
\texttt{\textbackslash xymatrix@dl{\ldots\ldots\ldots}} \texttt{\qquad} \texttt{\textbackslash xymatrix@dr{\ldots\ldots\ldots}} \$
```

will produce
Here are two more examples from \texttt{xyguide.ps}, to study, where the diagrams are rotated as well as scaled.

\[
\texttt{xymatrix}@r@C=1pc\
\ar[r]
\ar[d] & a'
\ar@{..>}[d] \\
\ar[rrr] & b & \ar@{..>}[r] & b'
\]

\[
\text{\left(}
\begin{array}{cc}
a & B \\
A & b
\end{array}
\right)
\]

\[
\begin{array}{cc}
a' & B' \\
A' & b'
\end{array}
\]