hibit n affinely independent permutations σ (and prove that they are affinely independent).)

Exercise 3-12. A *stable set* S (sometimes, it is called also an independent set) in a graph $G = (V, E)$ is a set of vertices such that there are no edges between any two vertices in S. If we let P denote the convex hull of all (incidence vectors of) stable sets of $G = (V, E)$, it is clear that $x_i + x_j \leq 1$ for any edge $(i, j) \in E$ is a valid inequality for P.

1. Give a graph G for which P is *not* equal to

$$\begin{aligned}\{x \in \mathbb{R}^{|V|} : \quad & x_i + x_j \leq 1 \quad && \text{for all } (i,j) \in E \\ & x_i \geq 0 && \text{for all } i \in V\}\end{aligned}$$

2. Show that if the graph G is bipartite then P equals

$$\begin{aligned}\{x \in \mathbb{R}^{|V|} : \quad & x_i + x_j \leq 1 \quad && \text{for all } (i,j) \in E \\ & x_i \geq 0 && \text{for all } i \in V\}.\end{aligned}$$

Exercise 3-13. Let $e_k \in \mathbb{R}^n$ ($k = 0, \ldots, n-1$) be a vector with the first k entries being 1, and the following $n-k$ entries being -1. Let $S = \{e_0, e_1, \ldots, e_{n-1}, -e_0, -e_1, \ldots, -e_{n-1}\}$, i.e. S consists of all vectors consisting of $+1$ followed by -1 or vice versa. In this problem set, you will study $\text{conv}(S)$.

1. Consider any vector $a \in \{-1, 0, 1\}^n$ such that (i) $\sum_{i=1}^n a_i = 1$ and (ii) for all $j = 1, \ldots, n-1$, we have $0 \leq \sum_{i=1}^j a_i \leq 1$. (For example, for $n = 5$, the vector $(1, 0, -1, 1, 0)$ satisfies these conditions.) Show that $\sum_{i=1}^n a_i x_i \leq 1$ and $\sum_{i=1}^n a_i x_i \geq -1$ are valid inequalities for $\text{conv}(S)$.
2. How many such inequalities are there?
3. Show that any such inequality defines a facet of $\text{conv}(S)$.

   (This can be done in several ways. Here is one approach, but you are welcome to use any other one as well. First show that either e_k or $-e_k$ satisfies this inequality at equality, for any k. Then show that the resulting set of vectors on the hyperplane are affinely independent (or uniquely identifies it).)
4. Show that the above inequalities define the entire convex hull of S.

   (Again this can be done in several ways. One possibility is to consider the 3rd technique described above.)

3.5 Total unimodularity

Definition 3.12 *A matrix A is totally unimodular (TU) if every square submatrix of A has determinant $-1, 0$ or $+1$.*

The importance of total unimodularity stems from the following theorem. This theorem gives a subclass of integer programs which are easily solved. A polyhedron P is said to be *integral* if all its vertices or extreme points are integral (belong to $\mathbb{Z}^n$).

Theorem 3.12 *Let A be a totally unimodular matrix. Then, for any integral right-hand-side b, the polyhedron*

$$P = \{x : Ax \leq b, x \geq 0\}$$

is integral.

Before we prove this result, two remarks can be made. First, the proof below will in fact show that the same result holds for the polyhedrons $\{x : Ax \geq b, x \geq 0\}$ or $\{x : Ax = b, x \geq 0\}$. In the latter case, though, a slightly weaker condition than totally unimodularity is sufficient to prove the result. Secondly, in the above theorem, one can prove the converse as well: If $P = \{x : Ax \leq b, x \geq 0\}$ is integral for all integral b then A must be totally unimodular (this is not true though, if we consider for example $\{x : Ax = b, x \geq 0\}$).

Proof: Adding slacks, we get the polyhedron $Q = \{(x, s) : Ax + Is = b, x \geq 0, s \geq 0\}$. One can easily show (see exercise below) that P is integral iff Q is integral.

Consider now any bfs of Q. The basis B consists of some columns of A as well as some columns of the identity matrix I. Since the columns of I have only one nonzero entry per column, namely a one, we can expand the determinant of A_B along these entries and derive that, in absolute values, the determinant of A_B is equal to the determinant of some square submatrix of A. By definition of totally unimodularity, this implies that the determinant of A_B must belong to $\{-1, 0, 1\}$. By definition of a basis, it cannot be equal to 0. Hence, it must be equal to ± 1.

We now prove that the bfs must be integral. The non-basic variables, by definition, must have value zero. The vector of basic variables, on the other hand, is equal to $A_B^{-1}b$. From linear algebra, A_B^{-1} can be expressed as

$$\frac{1}{\det A_B} A_B^{adj}$$

where A_B^{adj} is the adjoint (or adjugate) matrix of A_B and consists of subdeterminants of A_B. Hence, both b and A_B^{adj} are integral which implies that $A_B^{-1}b$ is integral since $|\det A_B| = 1$. This proves the integrality of the bfs. △

Exercise 3-14. Let $P = \{x : Ax \leq b, x \geq 0\}$ and let $Q = \{(x, s) : Ax + Is = b, x \geq 0, s \geq 0\}$. Show that x is an extreme point of P iff $(x, b - Ax)$ is an extreme point of Q. Conclude that whenever A and b have only integral entries, P is integral iff Q is integral.

In the case of the bipartite matching problem, the constraint matrix A has a very special structure and we show below that it is totally unimodular. This together with Theorem 3.12 proves Theorem 1.6 from the notes on the bipartite matching problem. First, let us restate the setting. Suppose that the bipartition of our bipartite graph is (U, V) (to avoid any confusion with the matrix A or the basis B). Consider

$$\begin{aligned} P &= \{x: & \sum_j x_{ij} = 1 & \qquad\qquad i \in U \\ & & \sum_i x_{ij} = 1 & \qquad\qquad j \in V \\ & & x_{ij} \geq 0 & \qquad\qquad i \in U, j \in V\} \\ &= \{x: & Ax = b, x \geq 0\}. & \end{aligned}$$

Theorem 3.13 *The matrix A is totally unimodular.*

The way we defined the matrix A corresponds to a *complete* bipartite graph. If we were to consider any bipartite graph then we would simply consider a submatrix of A, which is also totally unimodular by definition.

Proof: Consider any square submatrix T of A. We consider three cases. First, if T has a column or a row with all entries equal to zero then the determinant is zero. Secondly, if there exists a column or a row of T with only one $+1$ then by expanding the determinant along that $+1$, we can consider a smaller sized matrix T. The last case is when T has at least two nonzero entries per column (and per row). Given the special structure of A, there must in fact be *exactly* two nonzero entries per column. By adding up the rows of T corresponding to the vertices of U and adding up the rows of T corresponding to the vertices of V, one therefore obtains the same vector which proves that the rows of T are linearly dependent, implying that its determinant is zero. This proves the total unimodularity of A. △

We conclude with a technical remark. One should first remove one of the rows of A before applying Theorem 3.12 since, as such, it does not have full row rank and this fact was implicitly used in the definition of a bfs. However, deleting a row of A still preserves its totally unimodularity.

Exercise 3-15. If A is totally unimodular then A^T is totally unimodular.

Exercise 3-16. Use total unimodularity to prove König's theorem.

The following theorem gives a necessary and sufficient condition for a matrix to be totally unimodular.

Theorem 3.14 *Let A be a $m \times n$ matrix with entries in $\{-1, 0, 1\}$. Then A is TU if and only if for all subsets $R \subseteq \{1, 2, \cdots, n\}$ of rows, there exists a partition of R into R_1 and R_2 such that for all $j \in \{1, 2, \cdots, m\}$:*

$$\sum_{i \in R_1} a_{ij} - \sum_{i \in R_2} a_{ij} \in \{0, 1, -1\}.$$

We will prove only the *if* direction (but that is the most important as this allows to prove that a matrix is totally unimodular).

Proof: Assume that, for every R, the desired partition exists. We need to prove that the determinant of any $k \times k$ submatrix of A is in $\{-1, 0, 1\}$, and this must be true for any k. Let us prove it by induction on k. It is trivially true for $k = 1$. Assume it is true for $k - 1$ and we will prove it for k.

Let B be a $k \times k$ submatrix of A, and we can assume that B is invertible (otherwise the determinant is 0 and there is nothing to prove). The inverse B^{-1} can be written as $\frac{1}{\det(B)} B^*$, where all entries of B^* correspond to $(k-1) \times (k-1)$ submatrices of A. By our inductive hypothesis, all entries of B^* are in $\{-1, 0, 1\}$. Let b_1^* be the first row of B and e_1 be the k-dimensional row vector $[1\ 0\ 0 \cdots 0]$, thus $b_1^* = e_1 B^*$. By the relationship between B and B^*, we have that

$$b_1^* B = e_1 B^* B = \det(B) e_1 B^{-1} B = \det(B) e_1. \tag{5}$$

Let $R = \{i : b_{1i}^* \in \{-1, 1\}\}$. By assumption, we know that there exists a partition of R into R_1 and R_2 such that for all j:

$$\sum_{i \in R_1} b_{ij} - \sum_{i \in R_2} b_{ij} \in \{-1, 0, 1\}. \tag{6}$$

From (5), we have that

$$\sum_{i \in R} b_{1i}^* b_{ij} = \begin{cases} \det(B) & j = 1 \\ 0 & j \neq 1 \end{cases} \tag{7}$$

Since $\sum_{i \in R_1} b_{ij} - \sum_{i \in R_2} b_{ij}$ and $\sum_{i \in R} b_{1i}^* b_{ij}$ differ by a multiple of 2 for each j (since $b_{1i}^* \in \{-1, 1\}$), this implies that

$$\sum_{i \in R_1} b_{ij} - \sum_{i \in R_2} b_{ij} = 0 \quad j \neq 1. \tag{8}$$

For $j = 1$, we cannot get 0 since otherwise B would be singular (we would get exactly the 0 vector by adding and subtracting rows of B). Thus,

$$\sum_{i \in R_1} b_{i1} - \sum_{i \in R_2} b_{i1} \in \{-1, 1\}.$$

If we define $y \in \mathbb{R}^k$ by

$$y_i = \begin{cases} 1 & i \in R_1 \\ -1 & i \in R_2 \\ 0 & \textit{otherwise} \end{cases}$$

we get that $yB = \pm e_1$. Thus

$$y = \pm e_1 B^{-1} = \pm \frac{1}{\det B} e_1 B^* = \pm \frac{1}{\det B} b_1^*,$$

which implies that $\det B$ must be either 1 or -1. $\triangle$