Theorem 3.10 *If the face associated with $a_i^T x \leq b_i$ for $i \in I_<$ is not a facet then the inequality is redundant.*

And this one shows that facets are necessary:

Theorem 3.11 *If F is a facet of P then there must exists $i \in I_<$ such that the face induced by $a_i^T x \leq b_i$ is precisely F.*

In a *minimal* description of P, we must have a set of *linearly independent equalities* together with precisely one inequality for each facet of P.

Exercises

Exercise 3-6. Prove Corollary 3.7.

Exercise 3-7. Show that if $\text{rank}(A) < n$ then $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ has no vertices.

Exercise 3-8. Suppose $P = \{x \in \mathbb{R}^n : Ax \leq b, Cx \leq d\}$. Show that the set of vertices of $Q = \{x \in \mathbb{R}^n : Ax \leq b, Cx = d\}$ is a subset of the set of vertices of P.

(In particular, this means that if the vertices of P all belong to $\{0,1\}^n$, then so do the vertices of Q.)

Exercise 3-9. Given two extreme points a and b of a polyhedron P, we say that they are *adjacent* if the line segment between them forms an edge (i.e. a face of dimension 1) of the polyhedron P. This can be rephrased by saying that a and b are adjacent on P if and only if there exists a cost function c such that a and b are the only two extreme points of P minimizing $c^T x$ over P.

Consider the polyhedron (polytope) P defined as the convex hull of all perfect matchings in a (not necessarily bipartite) graph G. Give a necessary and sufficient condition for two matchings M_1 and M_2 to be adjacent on this polyhedron (hint: think about $M_1 \triangle M_2 = (M_1 \setminus M_2) \cup (M_2 \setminus M_1)$) and prove that your condition is necessary and sufficient.)

Exercise 3-10. Show that two vertices u and v of a polyhedron P are adjacent if and only there is a unique way to express their midpoint ($\frac{1}{2}(u+v)$) as a convex combination of vertices of P.

3.4 Polyhedral Combinatorics

In one sentence, polyhedral combinatorics deals with the study of polyhedra or polytopes associated with discrete sets arising from combinatorial optimization problems (such as matchings for example). If we have a discrete set X (say the incidence vectors of matchings in a graph, or the set of incidence vectors of spanning trees of a graph, or the set of incidence vectors of *stable* sets[1] in a graph), we can consider $\text{conv}(X)$ and attempt to describe it in terms

[1] A set S of vertices in a graph $G = (V, E)$ is stable if there are no edges between any two vertices of S.

of linear inequalities. This is useful in order to apply the machinery of linear programming. However, in some (most) cases, it is actually hard to describe the set of all inequalities defining $\text{conv}(X)$; this occurs whenever optimizing over X is hard and this statement can be made precise in the setting of computational complexity. For matchings, or spanning trees, and several other structures (for which the corresponding optimization problem is polynomially solvable), we will be able to describe their convex hull in terms of linear inequalities.

Given a set X and a proposed system of inequalities $P = \{x : Ax \leq b\}$, it is usually easy to check whether $\text{conv}(X) \subseteq P$. Indeed, for this, we only need to check that every member of X satisfies every inequality in the description of P. The reverse inclusion is more difficult. Here are 3 general techniques to prove that $P \subseteq \text{conv}(X)$ (if it is true!) (once we know that $\text{conv}(X) \subseteq P$).

1. **Algorithmically.** This involves linear programming duality. This is what we did in the notes about the assignment problem (minimum weight matchings in bipartite graphs). In general, consider any cost function c and consider the combinatorial optimization problem of maximizing $c^T x$ over $x \in X$. We know that:

$$\begin{aligned} \max\{c^T x : x \in X\} &= \max\{c^T x : x \in \text{conv}(X)\} \\ &\leq \max\{c^T x : Ax \leq b\} \\ &= \min\{b^T y : A^T y = c, y \geq 0\}, \end{aligned}$$

   the last equality coming from strong duality. If we can exhibit a solution $x \in X$ (say the incidence vector of a perfect matching in the assignment problem) and a dual feasible solution y (values u_i, v_j in the assignment problem) such that $c^T x = b^T y$ we will have shown that we have equality throughout, and if this is true for *any* cost function c, this implies that $P = \text{conv}(X)$.

   This is usually the most involved approach but also the one that works most often.

2. **Focusing on extreme points.** Show first that $P = \{x : Ax \leq b\}$ is bounded (thus a polytope) and then study its extreme points. If we can show that every extreme point of P is in X then we would be done since $P = \text{conv}(ext(P)) \subseteq \text{conv}(X)$, where $ext(P)$ denotes the extreme points of P (see Theorem 3.9). The assumption that P is bounded is needed to show that indeed $P = \text{conv}(ext(P))$ (not true if P is unbounded).

   In the case of the convex hull of bipartite matchings, this can be done easily and this leads to the notion of *totally unimodular* Matrices (TU), see the next section.

3. **Focusing on the facets of** $\text{conv}(X)$**.** This leads usually to the shortest and cleanest proofs. Suppose that our proposed P is of the form $\{x \in \mathbb{R}^n : Ax \leq b, Cx = d\}$. We have already argued that $\text{conv}(X) \subseteq P$ and we want to show that $P \subseteq \text{conv}(X)$.

   First we need to show that we are not missing any equality. This can be done for example by showing that $\dim(\text{conv}(X)) = \dim(P)$. We already know that $\dim(\text{conv}(X)) \leq \dim(P)$ (as $\text{conv}(X) \subseteq P$), and so we need to argue that $\dim(conv(X)) \geq \dim(P)$.

This means showing that if there are $n-d$ linearly independent rows in C we can find $d+1$ affinely independent points in X.

Then we need to show that we are not missing a valid inequality that induces a *facet* of $\text{conv}(X)$. Consider any valid inequality $\alpha^T x \leq \beta$ for $\text{conv}(X)$ with $\alpha \neq 0$. We can assume that α is any vector in $\mathbb{R}^n \setminus \{0\}$ and that $\beta = \max\{\alpha^T x : x \in \text{conv}(X)\}$. The face of $\text{conv}(X)$ this inequality defines is $F = \text{conv}(\{x \in X : \alpha^T x = \beta\})$. Assume that this is a non-trivial face; this will happen precisely when α is not in the row space of C. We need to make sure that if F is a facet then we have in our description of P an inequality representing it. What we will show is that if F is non-trivial then we can find an inequality $a_i^T x \leq b_i$ in our description of P such that (i) $F \subseteq \{x : a_i^T x = b_i\}$ and (ii) $a_i^T x \leq b_i$ defines a non-trivial face of P (this second condition is not needed if P is full-dimensional), or simply that every optimum solution to $\max\{\alpha^T x : x \in X\}$ satisfies $a_i^T x = b_i$, and that this inequality is not satisfied by all points in P. This means that if F was a facet, by maximality, we have a representative of F in our description.

This is a very simple and powerful technique, and this is best illustrated on an example.

Example. Let $X = \{(\sigma(1), \sigma(2), \cdots, \sigma(n)) : \sigma$ is a permutation of $\{1, 2, \cdots, n\}\}$. We claim that

$$\text{conv}(X) = \{x \in \mathbb{R}^n : \begin{array}{ll} \sum_{i=1}^n x_i = \binom{n+1}{2} & \\ \sum_{i \in S} x_i \geq \binom{|S|+1}{2} & S \subset \{1, \cdots, n\}\}. \end{array}$$

This is known as the *permutahedron*.

Here $\text{conv}(X)$ is not full-dimensional; we only need to show that we are not missing any facets and any equality in the description of $\text{conv}(P)$. For the equalities, this can be seen easily as it is easy to exhibit n affinely independent permutations in X. For the facets, suppose that $\alpha^T x \leq \beta$ defines a non-trivial facet F of $\text{conv}(X)$. Consider maximizing $\alpha^T x$ over all permutations x. Let $S = \arg\min\{\alpha_i\}$; by our assumption that F is non-trivial we have that $S \neq \{1, 2, \cdots, n\}$ (otherwise, we would have the equality $\sum_{i=1}^n x_i = \binom{n+1}{2}$). Moreover, it is easy to see (by an exchange argument) that any permutation σ whose incidence vector x maximizes $\alpha^T x$ will need to satisfy $\sigma(i) \in \{1, 2, \cdots, |S|\}$ for $i \in S$, in other words, it will satisfy the inequality $\sum_{i \in S} x_i \geq \binom{|S|+1}{2}$ at equality (and this is a non-trivial face as there exist permutations that do not satisfy it at equality). Hence, F is contained in a non-trivial face corresponding to an inequality in our description, and hence our description contains inequalities for all facets. This is what we needed to prove. That's it!

Exercises

Exercise 3-11. Consider the set $X = \{(\sigma(1), \sigma(2), \cdots, \sigma(n)) : \sigma$ is a permutation of $\{1, 2 \cdots, n\}\}$. Show that $\dim(\text{conv}(X)) = n-1$. (To show that $\dim(\text{conv}(X)) \geq n-1$, ex-