Only the Most Galvanizing Problems 3

On Wednesday May 31, there will be a quiz in class containing one of these questions.

1. Remember the second proof we saw in class for the bijection between $\mathbb{Q}$ and $\mathbb{Z}$. We listed the set $\mathbb{Q}^+$ of positive rationals by taking the rows of the infinite binary tree with $\frac{1}{1}$ on top of the tree, and with every node $\frac{i}{j}$ having two children: the left child being $\frac{i}{i+j}$ and the right child being $\frac{i+j}{j}$. Prove that
   (a) all fractions in the tree are reduced.
   (b) every reduced fraction $\frac{r}{s} > 0$ appears in the tree
   (c) every reduced fraction appears exactly once
   (d) the denominator of the nth fraction in the list equals the numerator of the $(n+1)$st.

2. Annie has decided to stop eating Clif bars because they are so expensive. To wean herself, she will never eat more Clif bars on one day than she did the day before. In total, she plans to eat n Clif bars before stopping altogether. For example, if $n = 10$, here are a few possibilities (among others):

$$8,2 \qquad 6,3,1 \qquad 5,3,2 \qquad 3,3,3,1 \qquad 2,2,2,2,2, \qquad 3,1,1,1,1,1,1,1$$

   Prove that the number of ways she can eat n Clif bars in at most k days, where $k \in \mathbb{Z}$, is equal to the number of ways she can eat n Clif bars with at most k Clif bars per day.

3. Suppose that an $n \times m$ board can be tiled with $1 \times k$ rectangular strips. Prove that n or m must be a multiple of k.

4. Let S be any set of ten distinct integers chosen from $\{1, 2, \ldots, 99\}$. Show that S always contains two disjoint subsets S_1 and S_2 such that $\sum_{i \in S_1} i = \sum_{i \in S_2} i$.