5. Matroid optimization

5.1 Definition of a Matroid

Matroids are combinatorial structures that generalize the notion of linear independence in matrices. There are many equivalent definitions of matroids, we will use one that focus on its *independent sets*. A matroid M is defined on a finite ground set E (or $E(M)$ if we want to emphasize the matroid M) and a collection of subsets of E are said to be *independent*. The family of independent sets is denoted by $\mathcal{I}$ or $\mathcal{I}(M)$, and we typically refer to a matroid M by listing its ground set and its family of independent sets: $M = (E, \mathcal{I})$. For M to be a matroid, $\mathcal{I}$ must satisfy two main axioms:

(I_1) if $X \subseteq Y$ and $Y \in \mathcal{I}$ then $X \in \mathcal{I}$,

(I_2) if $X \in \mathcal{I}$ and $Y \in \mathcal{I}$ and $|Y| > |X|$ then $\exists e \in Y \setminus X : X \cup \{e\} \in \mathcal{I}$.

In words, the second axiom says that if X is independent and there exists a larger independent set Y then X can be extended to a larger independent by adding an element of $Y \setminus X$. Axiom (I_2) implies that every *maximal* (inclusion-wise) independent set is maximum; in other words, all maximal independent sets have the same cardinality. A maximal independent set is called a *base* of the matroid.

Examples.

- One trivial example of a matroid $M = (E, \mathcal{I})$ is a **uniform** matroid in which
$$\mathcal{I} = \{X \subseteq E : |X| \leq k\},$$
for a given k. It is usually denoted as $U_{k,n}$ where $|E| = n$. A base is any set of cardinality k (unless $k > |E|$ in which case the only base is $|E|$).

  A **free** matroid is one in which all sets are independent; it is $U_{n,n}$.

- Another is a **partition** matroid in which E is partitioned into (disjoint) sets $E_1, E_2, \cdots, E_l$ and
$$\mathcal{I} = \{X \subseteq E : |X \cap E_i| \leq k_i \text{ for all } i = 1, \cdots, l\},$$
for some given parameters $k_1, \cdots, k_l$. As an exercise, let us check that (I_2) is satisfied. If $X, Y \in \mathcal{I}$ and $|Y| > |X|$, there must exist i such that $|Y \cap E_i| > |X \cap E_i|$ and this means that adding any element e in $E_i \cap (Y \setminus X)$ to X will maintain independence.

  Observe that M would *not* be a matroid if the sets E_i were *not* disjoint. For example, if $E_1 = \{1, 2\}$ and $E_2 = \{2, 3\}$ with $k_1 = 1$ and $k_2 = 1$ then both $Y = \{1, 3\}$ and $X = \{2\}$ have at most one element of each E_i, but one can't find an element of Y to add to X.

- **Linear** matroids (or representable matroids) are defined from a matrix A, and this is where the term *matroid* comes from. Let E denote the index set of the columns of A. For a subset X of E, let A_X denote the submatrix of A consisting only of those columns indexed by X. Now, define

  $$\mathcal{I} = \{X \subseteq E : \text{rank}(A_X) = |X|\},$$

  i.e. a set X is independent if the corresponding columns are linearly independent. A base B corresponds to a linearly independent set of columns of cardinality rank(A).

  Observe that (I_1) is trivially satisfied, as if columns are linearly independent, so is a subset of them. (I_2) is less trivial, but corresponds to a fundamental linear algebra property. If A_X has full column rank, its columns span a space of dimension $|X|$, and similarly for Y, and therefore if $|Y| > |X|$, there must exist a column of A_Y that is not in the span of the columns of A_X; adding this column to A_X increases the rank by 1.

  A linear matroid can be defined over any field $\mathbb{F}$ (not just the reals); we say that the matroid is **representable over** $\mathbb{F}$. If the field is $\mathbb{F}_2$ (field of 2 elements with operations (mod 2)) then the matroid is said to be **binary**. If the field is $\mathbb{F}_3$ then the matroid is said to be **ternary**.

  For example, the binary matroid corresponding to the matrix

  $$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

  corresponds to $U_{2,3}$ since the sum of the 3 columns is the 0 vector when taking components modulo 2. If A is viewed over the reals or over $\mathbb{F}_3$ then the matroid is the free matroid on 3 elements.

  Not every matroid is linear. Among those that are linear, some can be represented over some fields $\mathbb{F}$ but not all. For example, there are binary matroids which are not ternary and vice versa (for example, $U_{2,4}$ is ternary but not binary). Matroids which can be represented over *any* field are called **regular**. One can show that regular matroids are precisely those linear matroids that can be represented over the reals by a totally unimodular marix. (Because of this connection, a deep result of Seymour provides a polynomial-time algorithm for deciding whether a matrix is TU.)

- Here is an example of something that is not a matroid. Take a graph $G = (V, E)$, and let $\mathcal{I} = \{F \subseteq E : F \text{ is a matching}\}$. This is not a matroid since (I_2) is not necessarily satisfied ((I_1) is satisfied[1], however). Consider, for example, a graph on 4 vertices and let $X = \{(2,3)\}$ and $Y = \{(1,2), (3,4)\}$. Both X and Y are matchings, but one cannot add an edge of Y to X and still have a matching.

[1] When (I_1) alone is satisfied, $(E, \mathcal{I})$ is called an *independence system*.

- There is, however, another matroid associated with matchings in a (general, not necessarily bipartite) graph $G = (V, E)$, but this time the ground set of M corresponds to V. In the **matching matroid**, $\mathcal{I} = \{S \subseteq V : S \text{ is covered by some matching } M\}$. In this definition, the matching does not need to cover precisely S; other vertices can be covered as well.

- A very important class of matroids in combinatorial optimization is the class of **graphic** matroids (also called cycle matroids). Given a graph $G = (V, E)$, we define independent sets to be those subsets of edges which are forests, i.e. do not contain any cycles. This is called the graphic matroid $M = (E, \mathcal{I})$, or $M(G)$.

  (I_1) is clearly satisfied. To check (I_2), first notice that if F is a forest then the number of connected components of the graph (V, F) is given by $\kappa(V, F) = |V| - |F|$. Therefore, if X and Y are 2 forests and $|Y| > |X|$ then $\kappa(V, Y) < \kappa(V, X)$ and therefore there must exist an edge of $Y \setminus X$ which connects two different connected components of X; adding this edge to X results in a larger forest. This shows (I_2).

  If the graph G is connected, any base will correspond to a spanning tree T of the graph. If the original graph is disconnected then a base corresponds to taking a spanning tree in each connected component of G.

  A graphic matroid is a linear matroid. We first show that the field $\mathbb{F}$ can be chosen to be the reals. Consider the matrix A with a row for each vertex $i \in V$ and a column for each edge $e = (i, j) \in E$. In the column corresponding to (i, j), all entries are 0, except for a 1 in i or j (arbitrarily) and a -1 in the other. To show equivalence between the original matroid M and this newly constructed linear matroid M', we need to show that any independent set for M is independent in M' and vice versa. This is left as an exercise.

  In fact, a graphic matroid is *regular*; it can be represented over any field $\mathbb{F}$. In fact the above matrix A can be shown to be TU. To obtain a representation for a field $\mathbb{F}$, one simply needs to take the representation given above for $\mathbb{R}$ and simply view/replace all -1 by the additive inverse of 1.

5.1.1 Circuits

A minimal (inclusionwise) dependent set in a matroid is called a *circuit.* In a graphic matroid $M(G)$, a circuit will be the usual notion of a *cycle* in the graph G; to be dependent in the graphic matroid, one needs to contain a cycle and the minimal sets of edges containing a cycle are the cycles themselves. In a partition matroid, a circuit will be a set $C \subseteq E_i$ for some i with $|C \cap E_i| = k_i + 1$.

By definition of a circuit C, we have that if we remove any element of a circuit then we get an independent set. A crucial property of circuit is given by the following property,