Homework 6 - Math 409

In preparation of Quiz 6 on May 16

1. Show whether or not $M = (E(M), \mathcal{F}(M))$ is a matroid when
   - $E(M) = E$ for some graph $G = (V, E)$ and independent sets are matchings in G
   - $E(M) = V$ for some graph $G = (V, E)$ and independent sets are vertex covers in G
   - $E(M) = V$ for some graph $G = (V, E)$ and $\mathcal{F}(M) = \{S \subseteq V : \text{there exists a matching } M \text{ covering } S\}$
   - $E(M) = V$ for some graph $G = (V, E)$ and independent sets are stable sets of G
   - $E(M) = E_1 \cup \ldots \cup E_l$ where $E_1, \ldots, E_l$ are disjoint, and $\mathcal{F}(M) = \{S \subseteq E : |S \cap E_i| \leq k_i \ \forall i = 1, \ldots, l\}$ for some given constants $k_1, \ldots, k_l$
   - $E(M) = E_1 \cup \ldots \cup E_l$ where $E_1, \ldots, E_l$ are not necessarily disjoint, and $\mathcal{F}(M) = \{S \subseteq E : |S \cap E_i| \leq k_i \ \forall i = 1, \ldots, l\}$ for some given constants $k_1, \ldots, k_l$

2. One class of matroids we discussed in class is the class of *graphic* matroids, i.e. matroids where the ground set is composed of the edges of a graph $G = (V, E)$ and the independent sets are the edge sets of G-forests. We also discussed *linear* matroids, i.e. matroids where the ground set is composed of the indices of the columns of a matrix A and where we say a set of these indices is independent if the corresponding columns are linearly independent.
   (a) Show that any graphic matroid is also a linear matroid by constructing a matrix A where the rows are indexed by the vertices of V and the columns are indexed by the edges of E, and where a column vector indexed by (i, j) has 0's in every row, except for a 1 in the ith or jth row and a -1 in the other.
   (b) Show that any such matrix A is totally unimodular.

3. Let $M = (E, \mathcal{F})$ be a matroid. Let $k \in \mathbb{N}$ and define
$$\mathcal{F}_k = \{X \in \mathcal{F} : |X| \leq k\}.$$
   (a) Show that $M_k = (E, \mathcal{F}_k)$ is also a matroid.
   (b) What is the rank function of M_k if M has rank function r?

4. We are given n jobs that each take one unit of processing time. All jobs are available at time 0, and job j has a profit of c_j and a deadline d_j. The profit for job j will only be earned if the job completes by time d_j. The problem is to find an ordering of the jobs that maximizes the total profit. First, prove that if a subset of the jobs can be completed on time, then they can also be completed on time if they are scheduled in the order of their deadlines. Now, let $E(M) = \{1, 2, \ldots, n\}$ and let $\mathcal{F}(M) = \{S \subseteq E(M) : S \text{ can be completed on time}\}$. Prove that M is a matroid and describe how to find an optimal ordering for the jobs.