Homework 5 - Math 409

In preparation of Quiz 5 on May 9

1. In class, in the example for technique 3, we used the fact that $\dim(\text{conv}(X)) = n-1$ when

$$X = \{(\sigma(1), \sigma(2), \ldots, \sigma(n)) : \sigma \text{ is a permutation of } \{1, 2, \ldots, n\}\}.$$

Consider the family of permutations $\{\sigma_1, \ldots, \sigma_n\}$ given by $\sigma_i(1) = i$, $\sigma_i(i) = 1$ and $\sigma_i(j) = j$ (if $j \notin \{1, i\}$) for any $i \in [n]$. Show that those n permutations are affinely independent and how this implies that $\dim(\text{conv}(X)) = n-1$.

2. A stable set S in a graph $G = (V, E)$ is a set of vertices such that there are no edges between any two vertices in S. Let P denote the convex hull of all the incidence vectors of the stable sets of G. Clearly, $x_i + x_j \leq 1$ is a valid inequality for P for every edge $(i, j) \in E$.
   (a) Find a graph G for which P is not equal to

$$\begin{aligned}\{x \in \mathbb{R}^{|V|} : & x_i + x_j \leq 1 \ \forall (i, j) \in E \\ & x_i \geq 0 \ \forall i \in V\}\end{aligned}$$

   (b) Show that if the graph G is bipartite, then P is equal to the previous linear system. Use technique number 2. Do it in two ways: with and without using total unimodularity.

3. For $0 \leq k \leq n-1$, let $v_k \in \mathbb{R}^n$ be a vector where the first k entries are 1, and the following $n-k$ entries are -1. Let $S = \{v_0, v_1, \ldots, v_{n-1}, -v_0, -v_1, \ldots, -v_{n-1}\}$. Let $P = \text{conv}(S)$.
   (a) Show that $\sum_{i=1}^n a_i x_i \leq 1$ and $\sum_{i=1}^n a_i x_i \geq -1$ are valid inequalities for P when the following conditions on the a_i's hold:
      1. $a_i \in \{-1, 0, 1\}$ for all $i \in [n]$,
      2. $\sum_{i=1}^n a_i = 1$, and
      3. $0 \leq \sum_{i=1}^j a_i \leq 1$ for all $j \in [n-1]$.
   (b) How many such inequalities are there?
   (c) Consider any one of these inequalities: show that either v_k or $-v_k$ satisfies this inequality at equality for any k. Then show that all the tight $\pm v_k$'s are affinely independent. Finally show that the inequality is therefore a facet of P.
   (d) Use technique 3 to show that the above inequalities completely define P.

4. Suppose we have n activities to choose from. Activity i starts at time t_i and ends at time u_i. If chosen, activity i gives us a profit of p_i units. Our goal is to choose a subset of the activities that do not overlap (though an activity ending at time t and another one starting at time t is ok) and such that the total profit of the selected activities is maximum. **Hint:** how does this relate to problem 2?
   (a) Defining x_i as a variable that represents whether activity i is selected ($x_i = 1$) or not ($x_i = 0$), write an integer program of the form $\max\{p^\top x : Ax \leq b, x \in \{0, 1\}^n\}$ that would solve this problem.
   (b) Show that, if no more than two activities overlap at any given time, the matrix A is totally unimodular, implying that one can solve this problem by solving the linear program $\max\{p^\top x : Ax \leq b, 0 \leq x_i \leq 1 \ \forall i\}$.

5. Given a bipartite graph $G = (A \cup B, E)$ and given an integer k, let S_k be the set of all incidence vectors of matchings with at most k edges. Let

$$P_k = \{x : \sum_{j:(i,j)\in E} x_{ij} \leq 1 \ \forall i \in A$$
$$\sum_{i:(i,j)\in E} x_{ij} \leq 1 \ \forall j \in B$$
$$\sum_i \sum_j x_{ij} \leq k$$
$$x_{ij} \geq 0 \ \forall i \in A, j \in B\}.$$

(a) Without $\sum_i \sum_j x_{ij} \leq k$, we have shown that the resulting matrix is totally unimodular. Is it still with this additional constraint?

(b) Show that $P_k = \mathrm{conv}(S_k)$.