Homework 3 - Math 409

In preparation of Quiz 3 on April 18

1. Use the algorithm seen in class to find a maximum size matching (and check that it is maximum) in the graph below.

2. Give an example of a graph $G$, a matching $M$, and a blossom $B$ for $M$ such that a maximum matching in $G/B$ does not lead to a maximum matching in $G$. Explain why this does not contradict the following theorem that we proved in class:

   Let $B$ be a blossom with respect to $M$. Then $M$ is a maximum size matching in $G$ if and only if $M/B$ is a maximum size matching in $G/B$.

3. Let $G = (V, E)$ be any graph. Let $S \subseteq V$ be any set of vertices such that there exists a matching $M$ for which $S$ is a subset of the matched vertices in $M$. Prove that there exists a maximum matching $M^*$ for which all the vertices of $S$ are matched as well.

4. Let $G = (V, E)$ be any graph and let $U \subseteq V$ be a set of vertices such that there exists a matching $M$ of size $\frac{1}{2}(|U| + |V| - o(G\setminus U))$. Let $K_1, \ldots, K_l$ be the connected components of $G\setminus U$.

   (a) If $M$ is any maximum matching, then $M$ contains exactly $\left\lfloor \frac{|K_i|}{2} \right\rfloor$ edges from the subgraph of $G$ induced by the vertices of $K_i$.

   (b) If $M$ is any maximum matching, then each vertex $u \in U$ is matched to a vertex $v$ in an odd component $K_i$ of $G\setminus U$.

   (c) If $M$ is any maximum matching, then the only exposed vertices must be in odd components of $G\setminus U$.

5. Consider the Tutte-Berge formula $\max_M |M| = \min_{U \subseteq V} \frac{1}{2}(|U| + |V| - o(G\setminus U))$. Can there exist different sets $U \subseteq V$ that minimize the right handside? If so, find an example; otherwise, prove that the set $U$ that minimizes the right handside is unique.