Homework 2 - Math 409

Due on April 11, 2016

1. (10 points) (a) (4 points) Given the following table of preferences, find a stable matching using the stable marriage algorithm. Show your steps.

	Alice	Bea	Carol	Diane	Ed	Frank	George	Hon
1.	Ed	Ed	Ed	Hon	Alice	Bea	Bea	Alice
2.	Hon	Hon	George	George	Bea	Alice	Alice	Carol
3.	George	Frank	Frank	Frank	Carol	Diane	Diane	Bea
4.	Frank	George	Hon	Ed	Diane	Carol	Carol	Diane

   (b) (4 points) To find a perfect matching on $K_{n,n}$, one needs the constraints $\sum_j x_{ij} = 1$ for all $i \in [n]$ and $\sum_i x_{ij} = 1$ for all $j \in [n]$ where

$$x_{ij} = \begin{cases} 1 \text{ if edge } (i,j) \text{ is in the matching, and} \\ 0 \text{ otherwise} \end{cases}$$

   for all i, j. For the previous example, what constraint(s) can you add to get a stable matching? (Note: you can add new variables if you wish.)

   (c) (2 points) Write an objective function for the previous program that is meaningful (in any way you want). Explain its meaning. Will you get the same answer as in (a)?

2. (10 points) (a) (6 points) The Big Brothers and Big Sisters of America try to match m volunteers to n kids where $m > n$. You ask all of the volunteers to list the kids in order of compatibility, and you ask the kids to list the volunteers in the same way. Suppose you want to pair them in a stable way, i.e., there is not a kid and a volunteer who would rather be paired together than with whoever they got paired with. Adapt the stable marriage algorithm to do so. Prove that your algorithm terminates and that the result is stable.

   (b) (1 point) Who does your algorithm favor?

   (c) (3 points) What is the runtime of your algorithm?

3. (10 points) (a) (5 points) Using the Hungarian method, find a perfect matching of minimum weight in $K_{4,4}$ where $A = \{a, b, c, d\}$ and $B = \{1, 2, 3, 4\}$. Show your steps. The edge weights are in the following table.

	1	2	3	4
a	90	75	75	80
b	45	85	55	65
c	125	95	90	105
d	45	110	95	115

   (b) (2 points) Suppose the previous graph was missing some edges. How could you adapt the algorithm to still find a minimum cost perfect matching?

   (c) (3 points) Keeping the other edge weights the same, what can happen to the minimum cost when you remove an edge? Can it go up? Stay the same? Go down? Give examples of such an edge from the previous example whenever the answer is 'yes'; explain yourself whenever you answered 'no'.

4. (10 points) (a) (2 points) Use the greedy algorithm to find a perfect matching in the previous example. (The greedy example will repeatedly add the edge with minimum cost that is disjoint from all previously selected edges.)

   (b) (2 points) Find an example of a complete bipartite graph $G = (A \cup B, E)$ with $|A| = |B| = 2$ that shows that the value of the solution obtained by the greedy algorithm divided by the optimal solution can be arbitrarily large.

(c) (6 points) Let $G = (A \cup B, E)$ where $A = [n]$ and $B = [n]$ be a complete bipartite graph. Suppose that the vertices in A and in B are respectively associated to $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ where $a_i, b_j \in \mathbb{R}$ for all i, j. Let the weight on the edge (i, j) (with $i \in A$ and $j \in B$) be $b_j - a_i$ for all $i, j \in [n]$. Show that the solution obtained by the greedy algorithm in this case is (trivially) optimal.