Homework 1 - Math 409

Due on April 4, 2016

1. (10 points) Show that, given a bipartite graph \( G = (A \cup B, E) \), \( G \) has a matching of size \( |A| \) if and only if for every \( S \subseteq A \), we have \( |N(S)| \geq |S| \) where \( N(S) = \{b \in B : \exists a \in S \text{ with } (a, b) \in E\} \). **Hint:** one way to do so is to use the construction in the proof of König’s theorem.

2. (10 points) Consider the problem of perfectly tiling a subset of a checkerboard (like below) with dominoes (a domino being two adjacent squares).

   (a) (6 points) Show that this problem can be formulated as the problem of deciding whether a bipartite graph has a perfect matching.

   (b) (4 points) Can the following figure be tiled by dominoes? Give a tiling or a short proof that no tiling exists.

3. (10 points) A graph is \( d \)-regular if the degree of every vertex is \( d \), i.e., if the number of edges adjacent to any vertex is \( d \).

   (a) (6 points) Show that a \( d \)-regular bipartite graph has a perfect matching if \( d \geq 1 \). **Hint:** one way to do so is to use the statement you proved in the first question.

   (b) (3 points) Show that a \( d \)-regular bipartite graph has \( d \) disjoint perfect matchings. **Hint:** one way to do so is to use induction.

   (c) (1 point) Give an example of a \( d \)-regular non-bipartite graph that does not have a perfect matching.

4. (10 points) We say a matching \( M \) for \( G = (V, E) \) is **maximal** if \( M \cup e \) is not a matching for any \( e \in E \backslash M \). Show that in any (not necessarily bipartite) graph \( G = (V, E) \), the size of any maximal matching \( M \) is at least half the size of a maximum matching \( M^* \).