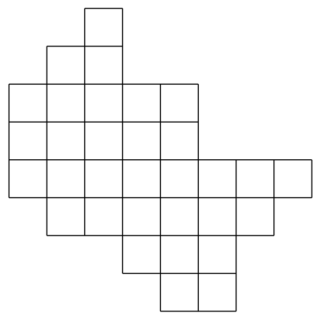
Homework 1 - Math 409

Due on April 4, 2016

1. (10 points) Show that, given a bipartite graph $G = (A \cup B, E)$, G has a matching of size $|A|$ if and only if for every $S \subseteq A$, we have $|N(S)| \geq |S|$ where $N(S) = \{b \in B : \exists a \in S \text{ with } (a, b) \in E\}$. **Hint:** one way to do so is to use the construction in the proof of König's theorem.

2. (10 points) Consider the problem of perfectly tiling a subset of a checkerboard (like below) with dominoes (a domino being two adjacent squares).
   (a) (6 points) Show that this problem can be formulated as the problem of deciding whether a bipartite graph has a perfect matching.
   (b) (4 points) Can the following figure be tiled by dominoes? Give a tiling or a **short** proof that no tiling exists.



3. (10 points) A graph is d-regular if the degree of every vertex is d, i.e., if the number of edges adjacent to any vertex is d.
   (a) (6 points) Show that a d-regular bipartite graph has a perfect matching if $d \geq 1$. **Hint:** one way to do so is to use the statement you proved in the first question.
   (b) (3 points) Show that a d-regular bipartite graph has d disjoint perfect matchings. **Hint:** one way to do so is to use induction.
   (c) (1 point) Give an example of a d-regular non-bipartite graph that does not have a perfect matching.

4. (10 points) We say a matching M for $G = (V, E)$ is **maximal** if $M \cup e$ is not a matching for any $e \in E \setminus M$. Show that in any (not necessarily bipartite) graph $G = (V, E)$, the size of any maximal matching M is at least half the size of a maximum matching M^*.