Deceptively Uninspiring Homework 5

Due Wednesday May 17th at the beginning of class

You may handwrite or type your answers/solutions/proofs. I highly encourage the use of a mathematical typesetting language (like LaTeX). If you handwrite, please make sure that your work is legible, and please staple your homework when you turn them in.

The Cantor-Schröder-Bernstein theorem states that if there exists an injective function $f : A \to B$ and an injective function $g : B \to A$, then A and B are equinumerous. It's a cool theorem. Perhaps one day you will see a proof. For now, feel free to use it in this assignment.

Also! I used $||S||$ for the cardinality of S in class the other day, but your textbook uses $|S|$, so let's use that instead.

1. Let S be the open interval of real numbers $(0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$. Prove that $|S| = |\mathbb{R}|$.

2. Let S be the open interval $(0, 1)$ and let T be the half-open interval $[0, 1) = \{x : 0 \leq x < 1\}$. Prove that $|S| = |T|$.

3. Let S be the set of all *finite* sets of natural numbers. (For example, $\{2, 5, 17, 200\} \in S$, but $\{n \in \mathbb{N} : n \text{ is even}\} \notin S$.). Prove that $|S| = |\mathbb{N}|$.

4. Define the Fibonacci sequence $\{F_n\}$ for $n \geq 0$ as follows: $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Prove that
$$\sum_{i=0}^{n} F_i = F_{n+2} - 1.$$

5. Prove using induction that the number of subsets of $[n] = \{1, \ldots, n\}$ is equal to 2^n, where n is a positive integer.