Annie's Survival Kit 6 - Math 324

1. (10 points) Use Green's theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \langle 2y + \cos(x), 4x - e^y \rangle$ and C is given by $\mathbf{r}(t) = \langle \cos(t) + 2, \sin(t) + 1 \rangle$ for $t \in [\frac{\pi}{4}, \frac{5\pi}{4}]$.

2. (10 points) (a) (7 points) Express the area of the region bounded by $\mathbf{r}(t) = \langle \sin(2t), \sin(t) \rangle$ for $0 \leq t \leq \pi$ with an integral of the form $\int_{t_0}^{t_1} f(t)\, dt$. **Do not evaluate**. Hint: think about how $\sin(2t)$ and $\sin(t)$ behave (which increases? decreases?) when $t \in [0, \frac{\pi}{4}]$, when $t \in [\frac{\pi}{4}, \frac{\pi}{2}]$, when $t \in [\frac{\pi}{2}, \frac{3\pi}{4}]$ and when $t \in [\frac{3\pi}{4}, \pi]$. Moreover, recall that the area of some region R is equal to $\int\int_R 1\, dA$. Finally, let D be some region and C its counterclockwise boundary, then Green's theorem states that, if $\mathbf{F}$ is defined and differentiable everywhere on D, then $\oint_C \mathbf{F} \cdot d\mathbf{r} = \int\int_D \text{curl}(\mathbf{F})\, dA$ where $\text{curl}(\mathbf{F}) = N_x - M_y$ for $\mathbf{F} = \langle M, N \rangle$.

   (b) (3 points) Using the following trigonometric identities, evaluate the integral you found in (a):

$$\sin^3(t) = \frac{3\sin(t) - \sin(3t)}{4}, \sin(2t) = 2\sin(t)\cos(t), \cos(2t) = \cos^2(t) - \sin^2(t) = 1 - 2\sin^2(t).$$

3. (10 points) Let $\mathbf{r}(t) = \langle 2\cos(t), \sin(2t) \rangle$ for $0 \leq t \leq \pi$. Express the mass of the region between $\mathbf{r}(t)$ and the x-axis with density equal to twice the distance from the y-axis with integrals of the form $\int_{t_0}^{t_1} f(t)\, dt$. Make sure $f(t)$ is free of absolute values. **Do not evaluate.**