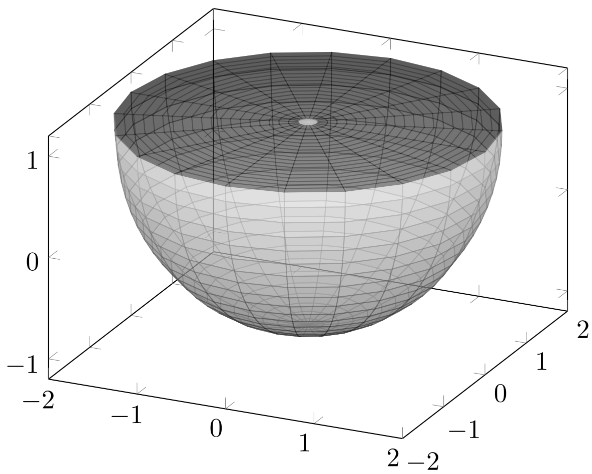
Annie's Survival Kit 2 - Math 324

1. (10 points) (a) (7 points) Switch the order of integration of $\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{-\sqrt{4-x^2-y^2}+1}^{1} 1\,dz\,dx\,dy$ to $dr\,d\theta\,dz$.

Answer: Since $z_{\min}(x,y) = -\sqrt{4-x^2-y^2}+1$ and $z_{\max} = 1$, the region is bounded below by the sphere $x^2+y^2+(z-1)^2 = 4$ (i.e. a sphere of radius two centered at $(0,0,1)$) and above by $z = 1$. Thus, the region is some part of the bottom hemisphere of the ball of radius two centered at $(0,0,1)$. To know which part, we must look at the projection of the region onto the xy-plane. Since $x_{\min}(y) = -\sqrt{4-y^2}$, $x_{\max}(y) = \sqrt{4-y^2}$, $y_{\min} = -2$ and $y_{\max} = 2$, the projection is the disk of radius two centered at the origin. Therefore, the region is the whole bottom hemisphere of the ball.



Switching to $dr\,d\theta\,dz$, we hit our region with planes $z = z_0$ for $-1 \leq z_0 \leq 1$. The intersection of $z = z_0$ and our half-ball is a full disk of radius $\sqrt{4-(z_0-1)^2}$ (since $x^2+y^2+(z-1)^2 = 4$ becomes $r^2+(z-1)^2 = 4$ in polar coordinates). Finally, $dz\,dx\,dy$ becomes $r\,dr\,d\theta\,dz$. Therefore, we obtain

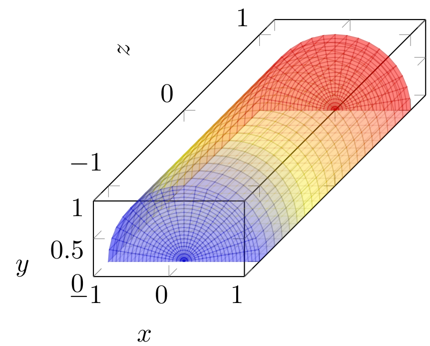
$$\int_{-1}^{1}\int_{0}^{2\pi}\int_{0}^{\sqrt{4-(z-1)^2}} r\,dr\,d\theta\,dz.$$

(b) (3 points) Knowing that $\int\int\int_R 1\,dV$ calculates the volume of a region R, solve the previous triple integral without doing any calculations.

Answer: Since $\int\int\int_R 1\,dV$ calculates the volume of R, the previous triple integral calculates the volume of a half-ball of radius two: $\frac{1}{2}\cdot\frac{4\pi 2^3}{3} = \frac{16\pi}{3}$.

2. (10 points) Switch the order of integration of $\int_0^\pi \int_0^1 \int_{-1}^1 zr^3\,dz\,dr\,d\theta$ to $dy\,dx\,dz$.

Answer: Since $z_{\min}(r,\theta) = -1$ and $z_{\max}(r,\theta) = 1$, the region is bounded below by the plane $z = -1$ and above by the plane $z = 1$. Since the projection onto the $r\theta$-plane (or, if you prefer, the xy-plane) is a half-disk of radius one that sits in $y \geq 0$, the region is a solid half-cylinder of height two (between $-1 \leq z \leq 1$) sitting in $y \geq 0$.



Switching to $dy\,dx\,dz$, fixing some x and z, I get a line that enters the region through the plane $y = 0$ and comes out on the cylinder where $y = \sqrt{1-x^2}$ (since the cylinder has equation $x^2 + y^2 = 1$). Projecting my half-cylinder onto the xz-plane, I get the square bounded by $-1 \leq x \leq 1$ and $-1 \leq z \leq 1$. Finally, since $dy\,dx\,dz = r\,dz\,dr\,d\theta$, we obtain

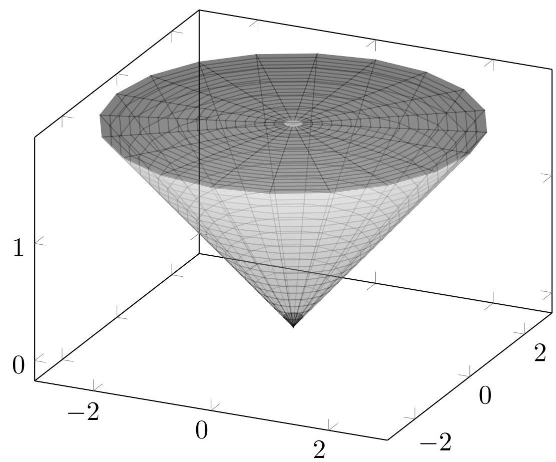
$$\int_{-1}^1 \int_{-1}^1 \int_0^{\sqrt{1-x^2}} z(x^2+y^2)\,dy\,dx\,dz.$$

3. (10 points) Consider a solid cone of height $\sqrt{3}$ with a 120° vertex angle. Its density at point P is equal to the distance from P to the central axis of the cone. Set up the integrals for the mass of the cone using cylindrical coordinates in two different orders: $dz\,dr\,d\theta$ and $dr\,d\theta\,dz$. Do not evaluate those integrals.

Hints:

- Choose and place the coordinate system to get the easiest integral possible.
- The mass of a solid region R with density δ is $\int\int\int_R \delta\,dV$.
- If the cone has a $\frac{2\pi}{3}$ vertex angle (the angle between its sides), what is the slope of its sides? How does the slope fit into the equation for a cone?

Answer: The easiest way to place the coordinate system is to put the vertex at the origin and center the cone opening up around the z-axis. Then the slope of the sides of the cone is $\frac{1}{\tan(\frac{\pi}{3})} = \frac{1}{\sqrt{3}}$. So the equation of the cone is $z = \frac{r}{\sqrt{3}}$.



The density is equal to the distance from the z-axis, so $\delta = r$.

Setting the triple integral with the order $dz\,dr\,d\theta$, we fix r and θ to obtain a vertical line parallel to the z-axis. If this line intersects the region, it first enters through the cone (where $z = \frac{r}{\sqrt{3}}$ and come out through the top of the cone (where $z = \sqrt{3}$). The projection of the region onto the $r\theta$-plane (or xy-plane), is a disk of radius 3, so the mass can be found by evaluating

$$\int_0^{2\pi}\int_0^3\int_{\frac{r}{\sqrt{3}}}^{\sqrt{3}} r^2\,dz\,dr\,d\theta.$$

Setting the triple integral with the order $dr\,d\theta\,dz$, we cut the region with planes $z = z_0$ for $0 \leq z_0 \leq \sqrt{3}$. The intersection of $z = z_0$ with our region is a disk of radius $\sqrt{3}z_0$. Therefore, the mass can also be found by evaluating

$$\int_0^{\sqrt{3}}\int_0^{2\pi}\int_0^{\sqrt{3}z} r^2\,dz\,dr\,d\theta.$$

4. (10 points) Set up a triple integral to find the volume of the region bounded by $z \leq x^2+y^2$, $x^2+y^2 \leq 3$ and $z \geq 0$ using **spherical coordinates**. (Recall that volume is $\int\int\int_R 1\,dV$.) **Do not evaluate.**

   Answer: The region is within the cylinder $x^2+y^2=3$: below the paraboloid $z=x^2+y^2$ and above the plane $z=0$. Therefore, fixing ϕ and θ, a half-line starting at the origin hits the paraboloid first (where $\rho = \frac{\cos(\phi)}{\sin^2(\phi)}$ since $z = x^2+y^2$ is $\rho\cos(\phi) = \rho^2\sin^2(\phi)$ in spherical coordinates) and then the cylinder (where $\rho = \frac{\sqrt{3}}{\sin(\phi)}$ since $x^2+y^2=3$ is $\rho^2\sin^2(\phi)=3$). The paraboloid and the cylinder intersect at $z=3$ in a circle of radius $\sqrt{3}$. Thus, $\phi_{\text{min}} = \tan^{-1}(\frac{\sqrt{3}}{3}) = \frac{\pi}{6}$, and since we are considering the region over $z=0$, $\phi_{\text{max}} = \frac{\pi}{2}$. Finally, θ is from 0 to 2π since we have a full revolution around the z-axis. Therefore, the volume is

$$\int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\int_{\frac{\cos(\phi)}{\sin^2(\phi)}}^{\frac{\sqrt{3}}{\sin(\phi)}} 1\rho^2\sin(\phi)\,d\rho d\phi d\theta.$$

5. (10 points) Switch $\displaystyle\int_0^{2\pi}\int_0^{\sqrt{3}}\int_2^3 zr^4\,dz\,dr\,d\theta + \int_0^{2\pi}\int_{\sqrt{3}}^2\int_2^{\sqrt{4-r^2}+2} zr^4\,dz\,dr\,d\theta$ to spherical coordinates.

   Answer: The region in the first triple integral is part of a solid cylinder of radius $\sqrt{3}$ centered around the z-axis of height 1 (with $2 \leq z \leq 3$). The region in the second triple integral is the part of the ball of radius two centered at $(0,0,2)$ (since $z = \sqrt{4-r^2}+2$) between $2 \leq z \leq 3$ and outside the aforementioned cylinder. Therefore, together, the region is the ball of radius two centered at $(0,0,2)$ cut with the planes $z=2$ and $z=3$.

   With spherical coordinates, fixing ϕ and θ, the half-line always enters through the plane $z=2$ (where $\rho = \frac{2}{\cos(\phi)}$), but either comes out on the sphere (where $\rho = 4\cos(\phi)$ since $x^2+y^2+(z-2)^2=4$ which is equivalent to $x^2+y^2+z^2-4z+4=4$ and thus $\rho^2 - 4\rho\cos(\phi) = 0$) or on the plane $z=3$ (where $\rho = \frac{3}{\cos(\phi)}$. We will thus need two triple integrals here too.

   When $\phi = 0$, we come out on $z=3$ and continue to do so until angle $\frac{\pi}{6}$. Then from $\frac{\pi}{6}$ to $\frac{\pi}{4}$, we come out on the sphere. Moreover, $0 \leq \theta \leq 2\pi$ since we have a full revolution around the z-axis.

   Finally, note that $zr^4\,dz\,dr\,d\theta = zr^3\,dV = \rho\cos(\phi)\rho^3\sin^3(\phi)\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta$. Thus, we obtain

$$\int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_{\frac{2}{\cos(\phi)}}^{\frac{3}{\cos(\phi)}} \rho^6\sin^4(\phi)\,d\rho\,d\phi\,d\theta + \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\int_{\frac{2}{\cos(\phi)}}^{4\cos(\phi)} \rho^6\sin^4(\phi)\,d\rho\,d\phi\,d\theta.$$

6. (10 points) Find the area of the ellipse $(2x+5y-7)^2 + (3x-7y+1)^2 \leq 1$.

   Answer: Let $u = 2x+5y-7$ and $v = 3x-7y+1$.

   The Jacobian is $\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ 3 & -7 \end{pmatrix} = -14-15 = -29$. Thus $dudv = |-29|dxdy$, so $\int\int_R 1\,dA$ becomes

$$\int\int_{u^2+v^2\leq 1} \frac{1}{29}\,du\,dv = \frac{1}{29}\pi 1^2 = \frac{\pi}{29}$$