Annie's Survival Kit 3 - Solutions - Math 324

1. (10 points) Set up a triple integral to find the volume of the region bounded by $z \leq x^2 + y^2$, $x^2 + y^2 \leq 3$ and $z \geq 0$ using **spherical coordinates**. (Recall that volume is $\int\int\int_R 1\, dV$.) **Do not evaluate.**

   Answer: The region is within the cylinder $x^2+y^2 = 3$: below the paraboloid $z = x^2+y^2$ and above the plane $z = 0$. Therefore, fixing ϕ and θ, a half-line starting at the origin hits the paraboloid first (where $\rho = \frac{\cos(\phi)}{\sin^2(\phi)}$ since $z = x^2 + y^2$ is $\rho\cos(\phi) = \rho^2 \sin^2(\phi)$ in spherical coordinates) and then the cylinder (where $\rho = \frac{\sqrt{3}}{\sin(\phi)}$ since $x^2+y^2 = 3$ is $\rho^2 \sin^2(\phi) = 3$). The paraboloid and the cylinder intersect at $z = 3$ in a circle of radius $\sqrt{3}$. Thus, $\phi_{\text{min}} = \tan^{-1}(\frac{\sqrt{3}}{3}) = \frac{\pi}{6}$, and since we are considering the region over $z = 0$, $\phi_{\text{max}} = \frac{\pi}{2}$. Finally, θ is from 0 to 2π since we have a full revolution around the z-axis. Therefore, the volume is

$$\int_0^{2\pi} \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_{\frac{\cos(\phi)}{\sin^2(\phi)}}^{\frac{\sqrt{3}}{\sin(\phi)}} 1\rho^2 \sin(\phi)\, d\rho d\phi d\theta.$$

2. (10 points) Switch $\displaystyle\int_0^{2\pi} \int_0^{\sqrt{3}} \int_2^3 zr^4\, dz\, dr\, d\theta + \int_0^{2\pi} \int_{\sqrt{3}}^2 \int_2^{\sqrt{4-r^2}+2} zr^4\, dz\, dr\, d\theta$ to spherical coordinates.

   Answer: The region in the first triple integral is part of a solid cylinder of radius $\sqrt{3}$ centered around the z-axis of height 1 (with $2 \leq z \leq 3$). The region in the second triple integral is the part of the ball of radius two centered at $(0, 0, 2)$ (since $z = \sqrt{4 - r^2}+2$) between $2 \leq z \leq 3$ and outside the aforementioned cylinder. Therefore, together, the region is the ball of radius two centered at $(0, 0, 2)$ cut with the planes $z = 2$ and $z = 3$.

   With spherical coordinates, fixing ϕ and θ, the half-line always enters through the plane $z = 2$ (where $\rho = \frac{2}{\cos(\phi)}$), but either comes out on the sphere (where $\rho = 4\cos(\phi)$ since $x^2 + y^2 + (z - 2)^2 = 4$ which is equivalent to $x^2 + y^2 + z^2 - 4z + 4 = 4$ and thus $\rho^2 - 4\rho\cos(\phi) = 0$) or on the plane $z = 3$ (where $\rho = \frac{3}{\cos(\phi)}$. We will thus need two triple integrals here too.

   When $\phi = 0$, we come out on $z = 3$ and continue to do so until angle $\frac{\pi}{6}$. Then from $\frac{\pi}{6}$ to $\frac{\pi}{4}$, we come out on the sphere. Moreover, $0 \leq \theta \leq 2\pi$ since we have a full revolution around the z-axis.

   Finally, note that $zr^4\, dz\, dr\, d\theta = zr^3\, dV = \rho\cos(\phi)\rho^3 \sin^3(\phi)\rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$. Thus, we obtain

$$\int_0^{2\pi} \int_0^{\frac{\pi}{6}} \int_{\frac{2}{\cos(\phi)}}^{\frac{3}{\cos(\phi)}} \rho^6 \cos(\phi) \sin^4(\phi)\, d\rho\, d\phi\, d\theta + \int_0^{2\pi} \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \int_{\frac{2}{\cos(\phi)}}^{4\cos(\phi)} \rho^6 \cos(\phi) \sin^4(\phi)\, d\rho\, d\phi\, d\theta.$$

3. (10 points) Find the area of the ellipse $(2x + 5y - 7)^2 + (3x - 7y + 1)^2 \leq 1$.

   Answer: Let $u = 2x + 5y - 7$ and $v = 3x - 7y + 1$.

   The Jacobian is $\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ 3 & -7 \end{pmatrix} = -14 - 15 = -29$. Thus $dudv = |-29|dxdy$, so $\int\int_R 1\, dA$ becomes

$$\int\int_{u^2+v^2 \leq 1} \frac{1}{29}\, du\, dv = \frac{1}{29}\pi 1^2 = \frac{\pi}{29}$$