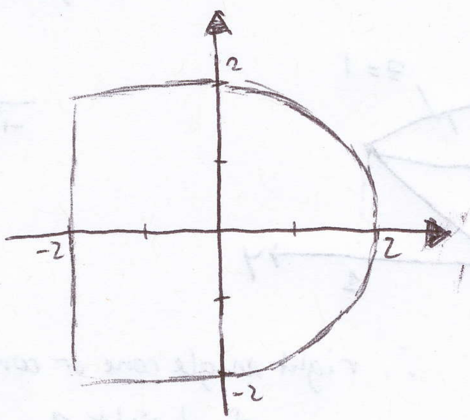
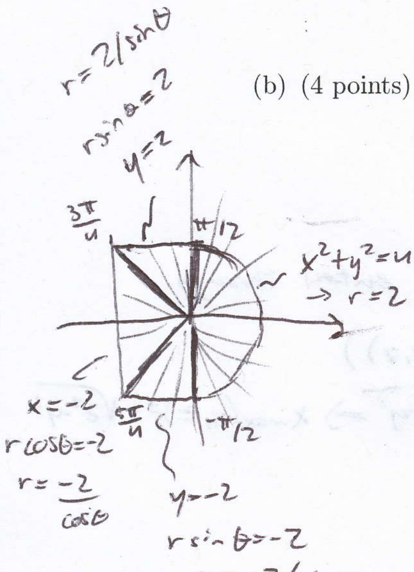
Annie's Survival Kit 2 - Math 324

1. (10 points) (a) (3 points) Draw the region of integration of $\int_{-2}^{2}\int_{-2}^{\sqrt{4-y^2}} x^2\,dx\,dy$.



(b) (4 points) Switch the previous double integral to polar coordinates. **Do not evaluate.**



$x^2+y^2=4$
$\to r=2$

$$\int_{-\pi/2}^{\pi/2}\int_0^2 r^3\cos^2\theta\,dr\,d\theta$$

$$+\int_{\pi/2}^{3\pi/4}\int_0^{2/\sin\theta} r^3\cos^2\theta\,dr\,d\theta$$

$$+\int_{3\pi/4}^{5\pi/4}\int_0^{-2/\cos\theta} r^3\cos^2\theta\,dr\,d\theta$$

$$+\int_{5\pi/4}^{3\pi/2}\int_0^{-2/\sin\theta} r^3\cos^2\theta\,dr\,d\theta$$

(c) (3 points) Suppose these integrals represent the moment of inertia around the y-axis ($\int\int_R x^2\delta\,dA$) for the region you found in (a). Describe with words what the density of the region is, and describe all the ways you can move the region and still obtain the same moment of inertia.

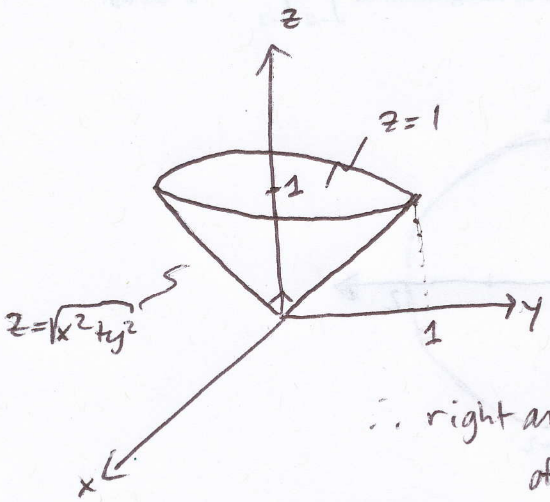
So $x^2\delta = x^2 \Rightarrow \delta = 1 \Rightarrow$ the density is constant

I can translate R up and down the y-axis since the contribution of each point depends only on x.

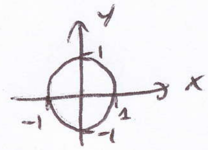
Moreover, I can flip the region across the y-axis since the contribution of x is the same as $-x$ $(x^2=(-x)^2)$.

I can of course also flip the region across the x-axis since I get the same region.

2. (10 points) (a) (3 points) Draw the region of integration of $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{1}|z|\,dz\,dy\,dx$.



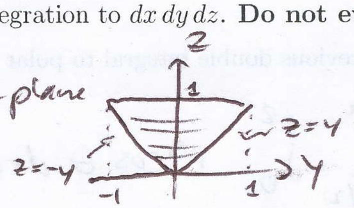
projection onto x,y-plane



$\therefore$ right angle cone or cone with slope 1 of height 1

(b) (4 points) Switch the order of integration to $dx\,dy\,dz$. **Do not evaluate.**

Projection onto yz-plane



Any line parallel to the x-axis that goes through R enters through the back of the cone ($z=\sqrt{x^2+y^2} \Rightarrow -\sqrt{z^2-y^2} = x_{min}(y,z)$) and comes out through the front of the cone ($z=\sqrt{x^2+y^2} \Rightarrow x_{max}(y,z) = +\sqrt{z^2-y^2}$)

$$\therefore \int_0^1\int_{-z}^{z}\int_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} |z|\,dx\,dy\,dz$$

(c) (3 points) Suppose these integrals represent the mass ($\int\int\int_R \delta\,dV$) of the region you found in (a). Describe with words what the density of the region is, and describe all the ways you can move the region and still obtain the same mass.

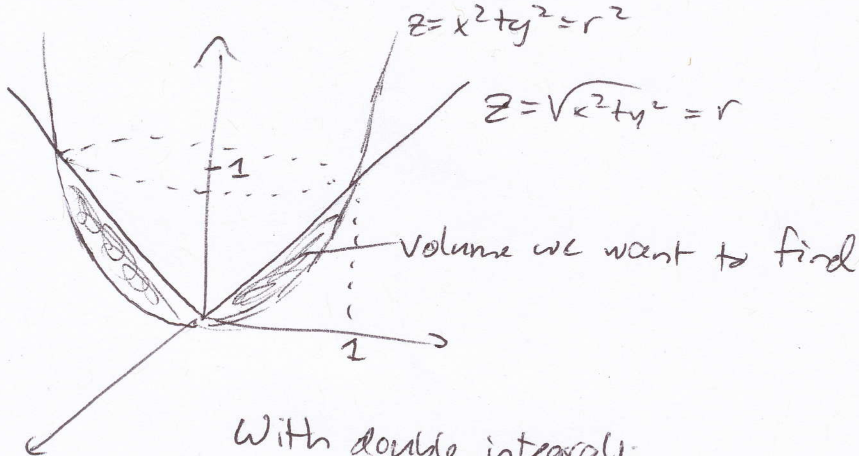
$\delta = |z| \Rightarrow \delta$ is the distance from the xy-plane

You can translate it anywhere on the xy-plane since doesn't change z

You can rotate it around its central axis since it doesn't change R

You can flip it across the xy-plane since $|z| = |-z|$

3. (10 points) Find the volume between $z = \sqrt{x^2+y^2}$ and $z = x^2+y^2$ over $R : x^2+y^2 \le 1$ in two ways: with double integrals and with triple integrals.



With double integrals:

Take volume below cone − volume below paraboloid

Polar is easiest

$$\int_0^{2\pi}\int_0^1 r \cdot r\,dr\,d\theta - \int_0^{2\pi}\int_0^1 r^2 \cdot r\,dr\,d\theta = \frac{2\pi}{3} - \frac{2\pi}{4} = \frac{\pi}{6}$$

With triple integrals: choosing z to be our inner variable, $z_{min}(x,y) = x^2+y^2$ and $z_{max}(x,y) = \sqrt{x^2+y^2}$ since any line parallel to the z-axis hits our 3D-region enters through the paraboloid and leaves through the cone. Projecting onto the xy-plane, we get the unit disk. Recall that the volume of D is $\iiint_D 1\,dV$.

So $$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} 1\,dz\,dy\,dx = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{x^2+y^2} - (x^2+y^2)\,dy\,dx$$

$$= \int_0^{2\pi}\int_0^1 (r-r^2)\,r\,dr\,d\theta = 2\pi\left(\frac{1}{3}-\frac{1}{4}\right) = \frac{\pi}{6}$$

switch to polar (or start with cylindrical)