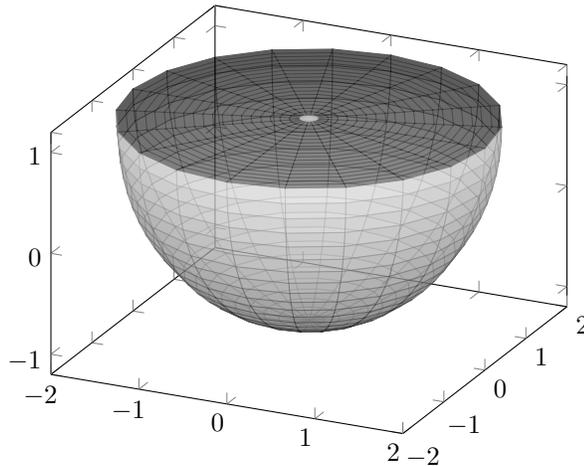


Annie's Survival Kit 2 - Math 324

1. (10 points) (a) (7 points) Switch the order of integration of $\int_{-2}^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{-\sqrt{4-x^2-y^2+1}}^1 1 \, dz \, dx \, dy$ to $dr \, d\theta \, dz$.

Answer: Since $z_{\min}(x, y) = -\sqrt{4-x^2-y^2} + 1$ and $z_{\max} = 1$, the region is bounded below by the sphere $x^2 + y^2 + (z-1)^2 = 4$ (i.e. a sphere of radius two centered at $(0, 0, 1)$) and above by $z = 1$. Thus, the region is some part of the bottom hemisphere of the ball of radius two centered at $(0, 0, 1)$. To know which part, we must look at the projection of the region onto the xy -plane. Since $x_{\min}(y) = -\sqrt{4-y^2}$, $x_{\max}(y) = \sqrt{4-y^2}$, $y_{\min} = -2$ and $y_{\max} = 2$, the projection is the disk of radius two centered at the origin. Therefore, the region is the whole bottom hemisphere of the ball.



Switching to $dr \, d\theta \, dz$, we hit our region with planes $z = z_0$ for $-1 \leq z_0 \leq 1$. The intersection of $z = z_0$ and our half-ball is a full disk of radius $\sqrt{4 - (z_0 - 1)^2}$ (since $x^2 + y^2 + (z-1)^2 = 4$ becomes $r^2 + (z-1)^2 = 4$ in polar coordinates). Finally, $dz \, dx \, dy$ becomes $r \, dr \, d\theta \, dz$. Therefore, we obtain

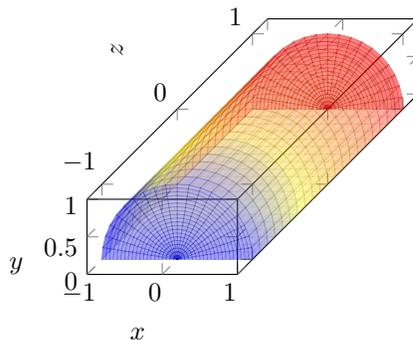
$$\int_{-1}^1 \int_0^{2\pi} \int_0^{\sqrt{4-(z-1)^2}} r \, dr \, d\theta \, dz.$$

- (b) (3 points) Knowing that $\iiint_R 1 \, dV$ calculates the volume of a region R , solve the previous triple integral without doing any calculations.

Answer: Since $\iiint_R 1 \, dV$ calculates the volume of R , the previous triple integral calculates the volume of a half-ball of radius two: $\frac{1}{2} \cdot \frac{4\pi 2^3}{3} = \frac{16\pi}{3}$.

2. (10 points) Switch the order of integration of $\int_0^\pi \int_0^1 \int_{-1}^1 zr^3 dz dr d\theta$ to $dy dx dz$.

Answer: Since $z_{\min}(r, \theta) = -1$ and $z_{\max}(r, \theta) = 1$, the region is bounded below by the plane $z = -1$ and above by the plane $z = 1$. Since the projection onto the $r\theta$ -plane (or, if you prefer, the xy -plane) is a half-disk of radius one that sits in $y \geq 0$, the region is a solid half-cylinder of height two (between $-1 \leq z \leq 1$) sitting in $y \geq 0$.



Switching to $dy dx dz$, fixing some x and z , I get a line that enters the region through the plane $y = 0$ and comes out on the cylinder where $y = \sqrt{1 - x^2}$ (since the cylinder has equation $x^2 + y^2 = 1$). Projecting my half-cylinder onto the xz -plane, I get the square bounded by $-1 \leq x \leq 1$ and $-1 \leq z \leq 1$. Finally, since $dy dx dz = r dz dr d\theta$, we obtain

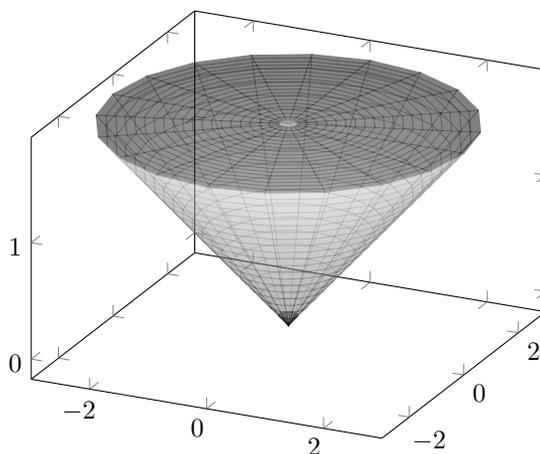
$$\int_{-1}^1 \int_{-1}^1 \int_0^{\sqrt{1-x^2}} z(x^2 + y^2) dy dx dz.$$

3. (10 points) Consider a solid cone of height $\sqrt{3}$ with a 120° vertex angle. Its density at point P is equal to the distance from P to the central axis of the cone. Set up the integrals for the mass of the cone using cylindrical coordinates in two different orders: $dz dr d\theta$ and $dr d\theta dz$. Do not evaluate those integrals.

Hints:

- Choose and place the coordinate system to get the easiest integral possible.
- The mass of a solid region R with density δ is $\int \int \int_R \delta dV$.
- If the cone has a $\frac{2\pi}{3}$ vertex angle (the angle between its sides), what is the slope of its sides? How does the slope fit into the equation for a cone?

Answer: The easiest way to place the coordinate system is to put the vertex at the origin and center the cone opening up around the z -axis. Then the slope of the sides of the cone is $\frac{1}{\tan(\frac{\pi}{3})} = \frac{1}{\sqrt{3}}$. So the equation of the cone is $z = \frac{r}{\sqrt{3}}$.



The density is equal to the distance from the z -axis, so $\delta = r$.

Setting the triple integral with the order $dz dr d\theta$, we fix r and θ to obtain a vertical line parallel to the z -axis. If this line intersects the region, it first enters through the cone (where $z = \frac{r}{\sqrt{3}}$ and come out through the top of the cone (where $z = \sqrt{3}$). The projection of the region onto the $r\theta$ -plane (or xy -plane), is a disk of radius 3, so the mass can be found by evaluating

$$\int_0^{2\pi} \int_0^3 \int_{\frac{r}{\sqrt{3}}}^{\sqrt{3}} r^2 dz dr d\theta.$$

Setting the triple integral with the order $dr d\theta dz$, we cut the region with planes $z = z_0$ for $0 \leq z_0 \leq \sqrt{3}$. The intersection of $z = z_0$ with our region is a disk of radius $\sqrt{3}z_0$. Therefore, the mass can also be found by evaluating

$$\int_0^{\sqrt{3}} \int_0^{2\pi} \int_0^{\sqrt{3}z} r^2 dz dr d\theta.$$