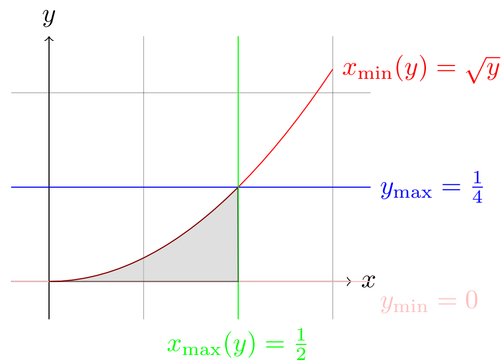
Annie's Survival Kit 1 - Math 324

1. (10 points) Evaluate $\int_0^{\frac{1}{4}} \int_{\sqrt{y}}^{\frac{1}{2}} \frac{e^x}{x}\, dx\, dy$ by changing the order of integration.

   Hint 1: first figure out what is the integration region R.
   Hint 2: recall that $\int u\, dv = uv - \int v\, du$.

   Answer: Note that $y_{\text{min}} = 0$, $y_{\text{max}} = \frac{1}{4}$, $x_{\text{min}}(y) = \sqrt{y}$ and $x_{\text{max}}(y) = \frac{1}{2}$. Thus, the region of integration is



   Thus, switching the way we slice, we get that $x_{\text{min}} = 0$, $x_{\text{max}} = \frac{1}{2}$, $y_{\text{min}}(x) = 0$ and $y_{\text{max}}(x) = x^2$. Therefore, our new double integral is

$$\int_0^{\frac{1}{2}} \int_0^{x^2} \frac{e^x}{x}\, dy\, dx.$$

   First solving the inner integral, we obtain

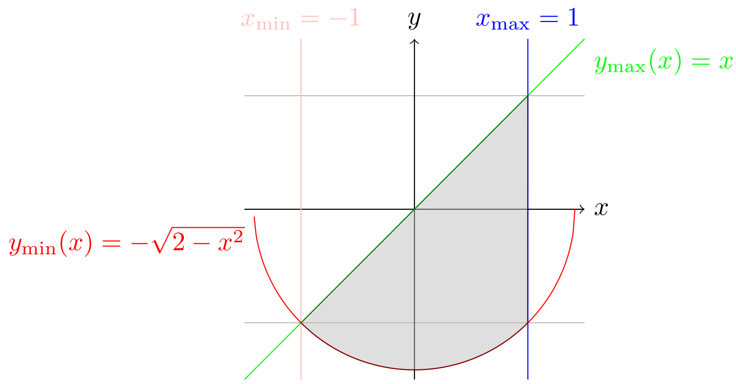
$$\left[\frac{e^x}{x} y\right]_0^{x^2} = xe^x.$$

   Thus, we obtain the outer integral $\int_0^{\frac{1}{2}} xe^x\, dx$. We need to use integration by parts: we set $u = x$, $dv = e^x dx$, so that $du = dx$, $v = e^x$. Thus, the outer integral is equal to

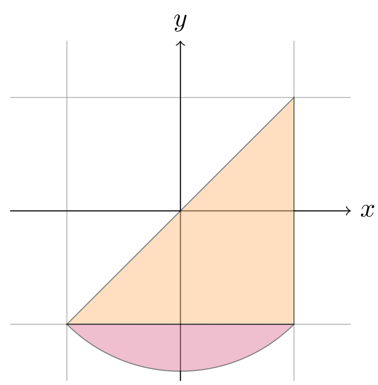
$$[xe^x]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} e^x\, dx = \frac{1}{2}e^{\frac{1}{2}} - (e^{\frac{1}{2}} - 1) = -\frac{1}{2}e^{\frac{1}{2}} + 1.$$

2. (10 points) (a) (5 points) Switch the order of integration of $\int_{-1}^{1}\int_{-\sqrt{2-x^2}}^{x} y\sqrt{x^2+y^2}\,dy\,dx$ to $dx\,dy$. Do not evaluate.

Answer: Note that $x_{\min} = -1$, $x_{\max} = 1$, $y_{\min}(x) = -\sqrt{2-x^2}$ and $y_{\max}(x) = x$. Thus, the region of integration is



Thus, switching the way we slice, we'll need two double integrals: one for R_1 and one for R_2.

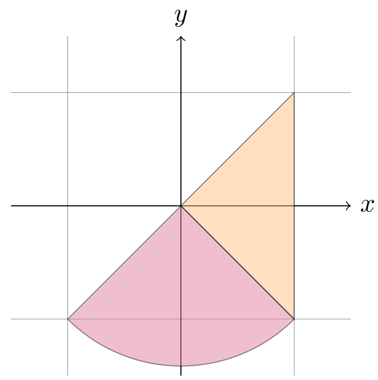


For R_1, we have that $y_{\min} = -1$, $y_{\max} = 1$, $x_{\min}(y) = y$ and $x_{\max}(y) = 1$. For R_2, we have that $y_{\min} = -\sqrt{2}$, $y_{\max} = -1$, $x_{\min}(y) = -\sqrt{2-y^2}$ and $x_{\max}(y) = \sqrt{2-y^2}$ Therefore, our new double integrals are

$$\int_{-1}^{1}\int_{y}^{1} y\sqrt{x^2+y^2}\,dx\,dy + \int_{-\sqrt{2}}^{-1}\int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}} y\sqrt{x^2+y^2}\,dx\,dy.$$

(b) (5 points) Switch $\int_{-1}^{1}\int_{-\sqrt{2-x^2}}^{x} y\sqrt{x^2+y^2}\,dy\,dx$ to polar coordinates. Do not evaluate.

Answer: Here again, our region splits into two: R_1 and R_2.



Here, R_1 is such that the $\theta_{\text{min}} = -\frac{\pi}{4}$, $\theta_{\text{max}} = \frac{\pi}{4}$, $r_{\text{min}}(\theta) = 0$ and $r_{\text{max}}(\theta) = \frac{1}{\cos(\theta)}$ since we exit on $x = 1$ (i.e. $r\cos(\theta) = 1$).

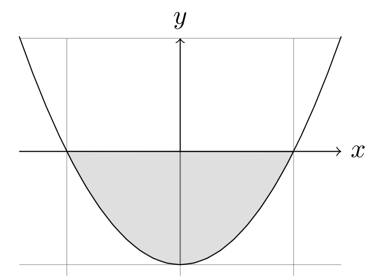
For R_2, we have $\theta_{\text{min}} = \frac{5\pi}{4}$, $\theta_{\text{max}} = -\frac{\pi}{4}$, $r_{\text{min}}(\theta) = 0$ and $r_{\text{max}}(\theta) = \sqrt{2}$ since we exit on the circle $x^2 + y^2 = 2$ (i.e. $r^2 = 2$).

Finally, note that $y\sqrt{x^2+y^2}$ becomes $r\sin(\theta)r$ and $dydx$ becomes $rdrd\theta$. Therefore, we obtain

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_{0}^{\frac{1}{\cos(\theta)}} r^3\sin(\theta)\,dr\,d\theta + \int_{\frac{5\pi}{4}}^{-\frac{\pi}{4}}\int_{0}^{\sqrt{2}} r^3\sin(\theta)\,dr\,d\theta$$

3. (10 points) (a) (7 points) Find the center of mass of a flat object with density proportional to the distance to the x-axis, and with region R bounded by $y = x^2 - 1$ and $y = 0$.

Answer: The region in question is



We can slice either vertically or horizontally. I'll choose the former. Then $x_{\min} = -1$, $x_{\max} = 1$, $y_{\min}(x) = x^2 - 1$ and $y_{\max}(x) = 0$. Moreover, the density is $\delta = k|y|$ for some constant k; since y is negative on our whole region, $\delta = -ky$. Thus, using the formula for the center of mass, we obtain

$$\bar{x} = \frac{\int_{-1}^{1} \int_{x^2-1}^{0} x \cdot (-ky)\, dy\, dx}{\int_{-1}^{1} \int_{x^2-1}^{0} -ky\, dy\, dx},$$

$$\bar{y} = \frac{\int_{-1}^{1} \int_{x^2-1}^{0} y \cdot (-ky)\, dy\, dx}{\int_{-1}^{1} \int_{x^2-1}^{0} -ky\, dy\, dx}.$$

Note that we can already notice that $\bar{x} = 0$ since the region is symmetric around the y-axis and the density is the same for all points at a same height y. Therefore, the contribution of any point (x, y) in the region will cancel out with the contribution of point $(-x, y)$ (which is also in the region).

For $\bar{y}$, we must actually evaluate the integrals. We have

$$\int_{-1}^{1} \int_{x^2-1}^{0} -ky^2\, dy\, dx = -k \int_{-1}^{1} \left[\frac{y^3}{3} \right]_{x^2-1}^{0} dx = \frac{k}{3} \int_{-1}^{1} x^6 - 3x^4 + 3x^2 - 1\, dx = \frac{-32k}{105}$$

and

$$\int_{-1}^{1} \int_{x^2-1}^{0} -ky\, dy\, dx = -k \int_{-1}^{1} \left[\frac{y^2}{2} \right]_{x^2-1}^{0} dx = \frac{k}{2} \int_{-1}^{1} x^4 - 2x^2 + 1\, dx = \frac{8k}{15}.$$

Therefore, $\bar{y} = -\frac{4}{7}$.

(b) (3 points) Without doing further calculations, find the center of mass of a flat object with density proportional to the distance to the line $y = 3$, and with region R bounded by $y = x^2 + 2$ and $y = 3$.

Answer: Note that both the region and density were shifted up by three units. Therefore, the center of mass also shifts up by three units to become $(0, -\frac{4}{7} + 3) = (0, \frac{17}{7})$.