Math 308L
Autumn 2016
Midterm 2
November 18, 2016
Time Limit: 50 Minutes

Name (Print): ____________________

Instructor ____________________

This exam contains 6 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use a sheet of notes, 8.5×11 and handwritten by you, but no other devices, books, or notes are permitted.

Unless otherwise stated, you are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	4	
3	10	
4	10	
5	6	
Total:	40	

1. (a) (6 points) Let I denote the 3×3 identity matrix, and suppose A and B are 3×3 invertible matrices with $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 7 \\ 8 & -1 & 3 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
Find a 3×3 matrix X that satisfies the equation

$$AXB - AB = I.$$

Solution: First we solve for X.

$$\begin{aligned} & AXB - AB = I \\ \implies\ & AXB = I + AB \\ \implies\ & XB = A^{-1} + B \\ \implies\ & X = A^{-1}B^{-1} + I \end{aligned}$$

We then just compute $A^{-1}B^{-1} + I = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 9 & 7 \\ 6 & 4 & 12 \end{bmatrix}$.

Comments: The identity matrix I is the *multiplicative* identity. So if you add it to something, it does not go away. Also, order matters. You can multiply both sides of an equation by A^{-1}, but you have to be consistent about whether you multiply on the left or right.

(b) (4 points) Suppose $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \end{bmatrix}$. Compute the rank and nullity of A.

If you put A into echelon form, it has 2 nonzero rows, and so the row space of A has dimension 2. Therefore $\text{rank}(A) = 2$, and so by the rank nullity theorem $\text{nullity}(A) = 5 - 2 = 3$.

2. Decide whether the following statements are true or false, and circle the explanation that best explains why.

(a) (2 points) If A and B are invertible $n \times n$ matrices, then $A+B$ is invertible.

1. True. You can distribute the inverse, so that it becomes $(A+B)^{-1} = (A^{-1} + B^{-1})$.
2. True. If you row reduce A or B you get the identity matrix, so if you row reduce $A+B$ you get $2I$ which is equivalent to the identity.
3. **False**. For example, let $A = I$ and $B = -I$. Then $A+B$ is the matrix of all zeros and hence is not invertible.
4. False. Consider the 2×2 matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$.

(b) (2 points) If A is a square matrix and $A^T A = I$, then A must be invertible.

1. **True**. If $A\mathbf{x} = \mathbf{0}$, then $\mathbf{x} = A^T A\mathbf{x} = \mathbf{0}$, so $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
2. True. Consider for example the matrix $A = I$. We see that $I^T I = I$.
3. False. Consider for example $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.
4. False. The product of a matrix and its transpose can never be the identity matrix.

3. Let A be the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) (6 points) Compute the inverse of A.

$$A^{-1} = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) (4 points) Compute the inverse of A^2.

$(A^2)^{-1} = (A \cdot A)^{-1} = A^{-1} \cdot A^{-1}$, and so we compute $(A^2)^{-1} = \begin{bmatrix} 1 & -4 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$.

4. (a) (5 points) Give an example of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ that sends $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Is the linear transformation you chose one-to-one? Onto? Explain.

Consider for example the linear transformation $T(x) = Ax$, where $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$. Then T is not one-to-one, because the columns of A are linearly dependent. T is also not onto - in fact no linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ can be onto, since $2 < 3$.
(Of course there are a lot of other possible solutions here.)

Comments: A lot of people said their linear transformation was one-to-one because $m \leq n$. This is not sufficient! For example the one I gave above is not one-to-one. (It is possible for it to be one-to-one when $m \leq n$, but not necessary. Depends on the actual linear transformation.)

(b) (5 points) Suppose A is an $n \times m$ matrix whose columns span $\mathbb{R}^n$. Decide which of the following must be true. Circle any that must be true.

1. $\mathbf{m \geq n}$.
2. The columns of A are linearly dependent.
3. **The nullity of A is $m - n$.**
4. The system $A\mathbf{x} = \mathbf{b}$ has exactly one solution for any $\mathbf{b} \in \mathbb{R}^n$.
5. **The dimension of row(A) is n.**

5. (6 points) Choose **ONE OF THE FOLLOWING** to prove or disprove. Make it very clear which you are choosing.

   (a) Prove or disprove: If S_1 and S_2 are subspaces of $\mathbb{R}^n$, then their intersection $S = S_1 \cap S_2$ is also a subspace of $\mathbb{R}^n$. (Recall that the intersection of two sets is the set of all elements that are contained in both sets.)

   (b) Prove or disprove: If S_1 and S_2 are subspaces of $\mathbb{R}^n$, then their union $S = S_1 \cup S_2$ is also a subspace of $\mathbb{R}^n$. (Recall that the union of two sets is the set of all elements that are contained in either set.)

Proof of (a): First note that $S \subset \mathbb{R}^n$ since both S_1 and S_2 are. Since $0 \in S_1$ and $0 \in S_2$, we know $0 \in S$. Now suppose that $u, v \in S$. Then $u, v \in S_1$, and so $u + v \in S_1$ since S_1 a subspace. Similarly, u and v must be in S_2, and so $u + v \in S_2$ since S_2 is a subspace. Therefore $u + v \in S_1 \cap S_2 = S$.

Now suppose that $r \in \mathbb{R}$ and $u \in S$. Then $u \in S_1$, and so $ru \in S_1$ since S_1 a subspace. Similarly, u must be in S_2, and so $ru \in S_2$ since S_2 is a subspace. Therefore $ru \in S_1 \cap S_2 = S$.

Since u and v were arbitrary elements of S and r an arbitrary number, S satisfies the three conditions of being a subspace and hence is a subspace of $\mathbb{R}^n$.

Disproving (b): Consider the subspaces of $\mathbb{R}^2$ $S_1 = \text{span}\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$ and $S_1 = \text{span}\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\}$. (Notice that S_1 and S_2 correspond to the x and y axes.) Both S_1 and S_2 are subspaces, but their union is not. For instance, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ are in $S_1 \cup S_2$, but their sum $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not in $S_1 \cup S_2$.

Comments: When you prove things, I expect you to prove them in general and not just for a specific example. That said, it can be really helpful to think about some examples and see what happens for them. This can help you figure out whether the thing you want to prove is true, it can help you find counterexamples, and it can help you figure out how to write out a more general proof.