Math 308O **Name (Print):** ______________
Winter 2015
Final Exam
March 16, 2015
Time Limit: 1 Hour 50 Minutes Instructor ______________

This exam contains 10 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

For question 1 (to be completed separately in the first ten minutes and handed in) you may not use any notes or calculators. For questions 2-7, you may use a scientific calculator and a sheet of notes, 8.5×11 and handwritten by you, but no other devices, books, or notes are permitted.
You are required to show your work on each problem on this exam, unless otherwise specified. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	15	
2	15	
3	15	
4	10	
5	15	
6	21	
7	9	
Total:	100	

1. (15 points) Suppose S is a subspace of $\mathbb{R}^n$. Prove that the orthogonal complement of S, $S^\perp$, is also a subspace of $\mathbb{R}^n$.

Solution: First note that $\mathbf{0} \in \mathbf{S}^\perp$ since $\mathbf{0}$ is orthogonal to everything. Therefore the first condition of being a subspace is satisfied.

Now suppose $\mathbf{u}, \mathbf{v} \in \mathbf{S}^\perp$. Then for any $\mathbf{s} \in \mathbf{S}$, $\mathbf{u} \cdot \mathbf{s} = \mathbf{0}$ and $\mathbf{v} \cdot \mathbf{s} = \mathbf{0}$ by definition of $S^\perp$. But then $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{s} = \mathbf{u} \cdot \mathbf{s} + \mathbf{u} \cdot \mathbf{s} = \mathbf{0} + \mathbf{0} = \mathbf{0}$. Since $\mathbf{s}$ was an arbitrary element of S, we have proved that $\mathbf{u} + \mathbf{v} \in \mathbf{S}^\perp$. This holds for any $\mathbf{u}, \mathbf{v} \in \mathbf{S}^\perp$, and so the second condition of being a subspace is satisfied.

Finally, suppose $\mathbf{u} \in \mathbf{S}^\perp$ and $r \in \mathbb{R}$. Then for any $\mathbf{s} \in \mathbf{S}$, $\mathbf{u} \cdot \mathbf{s} = \mathbf{0}$ by definition of $S^\perp$. But then $(r\mathbf{u}) \cdot \mathbf{s} = \mathbf{ru} \cdot \mathbf{s} = \mathbf{r0} = \mathbf{0}$. Since $\mathbf{s}$ was an arbitrary element of S, we have proved that $r\mathbf{u} \in \mathbf{S}^\perp$. This holds for any $\mathbf{u} \in \mathbf{S}^\perp$ and $r \in \mathbb{R}$, and so the third condition of being a subspace is satisfied.

2. Suppose we have

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

(a) (3 points) Is $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ an orthogonal set? Justify your answer.

Solution: No. For example, $\mathbf{v_2}$ and $\mathbf{v_3}$ are not orthogonal. (Their dot product is 1.)

(b) (4 points) Is $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ a basis for $\mathbb{R}^3$? Justify your answer.

Solution: Yes. We can compute that the determinant of $[v_1 v_2 v_3]$ is 3, which is not zero.

(c) (5 points) Use projections to find an orthogonal basis for $\mathbb{R}^3$ that includes the vectors $\mathbf{v_1}$ and $\mathbf{v_2}$. That is, you must find a vector $\mathbf{w}$ so that $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{w}\}$ is an orthogonal basis. (Remember to check your work.)

Basically we want to use Gram-Schmidt here, but if we do Gram-Schmidt to get the vectors w_1, w_2, w_3, we see that $w_1 = v_1$ and $w_2 = v_2$. So we only have to worry about the third vector. We get $w = w_3 = v_3 - \text{proj}_{w_2} v_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{bmatrix}$. We can check that this is orthogonal to v_1 and v_2, and so the set $\{v_1, v_2, w\}$ is an orthogonal basis, as desired.

(d) (3 points) What is $\text{proj}_S(\mathbf{w})$, where $S = \text{span}\{\mathbf{v_1}, \mathbf{v_2}\}$?

w is orthogonal to v_1 and v_2, so its projection onto S is zero.

3. Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) (5 points) Find the characteristic polynomial and all eigenvalues of the matrix A.

The characteristic polynomial is $p(\lambda) = (1-\lambda)(\lambda^2 - 1)$. Setting this equal to zero tells us the eigenvalues are ± 1.

(b) (7 points) Choose an eigenvalue λ of the matrix A. (It doesn't matter which one you pick.) For this eigenvalue, find a basis for the corresponding eigenspace.

You can do this problem in the normal way, where we solve the system $(A - \lambda I)x = 0$. For $\lambda = 1$, we get a basis of $\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \}$. For $\lambda = -1$, we get a basis of $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

Remark: There is actually an interesting way to do this problem with less work. You can notice that what A does to a vector is it switches the first two components. (Think about this a little bit.) So how do you get it to stay put? You have the first two components be the same. How do you get it to multiply by -1? You have the first two components be negatives of each other. If this doesn't make sense don't worry about it, but it's kind of interesting.

(c) (3 points) For the eigenvalue you chose in part (b), does the multiplicity of the eigenvalue equal the dimension of the corresponding eigenspace? Yes.

4. (a) (5 points) Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. Is A^3 invertible? Explain. If it is, find the determinant of the inverse of A^3. In other words, find $\det((A^3)^{-1})$.

Notice that A is invertible because $\det(A) = 4$. Therefore A^3 is invertible with inverse $(A^3)^{-1} = (A^{-1})^3$. Then using properties of determinants, we get $\det((A^3)^{-1}) = \frac{1}{4^3} = \frac{1}{64}$

(b) (5 points) Suppose A, B, and $A + B$ are invertible matrices. Simplify the following expression as much as possible.

$$(A + B)^T (A^T + B^T)^{-1} B(A + B)^{-1} B^{-1}.$$

We can simplify this to $B(A + B)^{-1} B^{-1}$.

5. (a) (5 points) Let $\mathcal{S}$ be the standard basis, and suppose $\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$. If $x_{\mathcal{B}_1} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}_{\mathcal{B}_1}$, find $x_{\mathcal{S}}$.

$$x_{\mathcal{S}} = \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}.$$

(b) (5 points) Now consider the basis $\mathcal{B}_2 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$. If $x_{\mathcal{S}} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}_{\mathcal{S}}$, find $x_{\mathcal{B}_2}$.

$$x_{\mathcal{B}_2} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{\mathcal{B}_2}.$$

(c) (5 points) If $x_{\mathcal{B}_1} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}_{\mathcal{B}_1}$, compute $x_{\mathcal{B}_2}$.

$$x_{\mathcal{B}_2} = \begin{bmatrix} 3 \\ 6 \\ 5 \end{bmatrix}_{\mathcal{B}_2}.$$

6. Read each of the following statements carefully, and decide whether it is true or false. You are not required to justify your answers, but I recommend justifying them to yourself.

   (a) (3 points) If the columns of A form an orthonormal basis for $\mathbb{R}^n$, then $A^T = A^{-1}$.

   True.

   (b) (3 points) Suppose $\mathbf{x_1}$ and $\mathbf{x_2}$ are vectors in $\mathbb{R}^m$, and A is an $n \times m$ matrix. If $\mathbf{x_1}$ and $\mathbf{x_2}$ are both solutions to the equation $A\mathbf{x} = \mathbf{0}$, then so is the vector $\mathbf{x_1} + \mathbf{x_2}$.

   True.

   (c) (3 points) The set $\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 15 \\ 0 \\ 0 \end{bmatrix} \right\}$ spans all of $\mathbb{R}^4$.

   False.

   (d) (3 points) Suppose $\hat{\mathbf{x}}$ in $\mathbb{R}^m$ is a least squares solution to the system $A\mathbf{x} = \mathbf{y}$. Then $\|A\hat{\mathbf{x}} - \mathbf{y}\| \leq \|\mathrm{A}\mathbf{x} - \mathbf{y}\|$ for any $\mathbf{x}$ in $\mathbb{R}^m$.

   True.

(e) (3 points) If λ is an eigenvalue for A, then λ^2 is an eigenvalue of A^2.

True.

(f) (3 points) The function $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_3 + 1 \end{bmatrix}$ is a linear transformation.

False.

(g) (3 points) Suppose $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is a linearly independent set of vectors, and we run the Gram-Schmidt process on them to obtain an orthogonal set of vectors $\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$. Then $\text{span}\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\} = \text{span}\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$.

True.

7. (a) (3 points) Give an example of an orthogonal set of 5 vectors in $\mathbb{R}^5$ that does not span $\mathbb{R}^5$.

$\{0, e_1, e_2, e_3, e_4\}$.

(b) (6 points) Find a value for k so that the set $\left\{ \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ k \end{bmatrix} \right\}$ is linearly dependent.

$k = -\frac{1}{2}$.