Math 308L
Autumn 2016
Final Exam
December 15, 2016
Time Limit: 110 Minutes

Name (Print): ____________________

Instructor ____________________

This exam contains 9 pages (including this cover page) and 8 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use a sheet of notes, 8.5×11 and handwritten by you, but no other devices, books, or notes are permitted.

Unless otherwise stated, you are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	12	
3	12	
4	10	
5	10	
6	6	
7	10	
8	10	
Total:	80	

1. Let V denote the set of 4 vectors in $\mathbb{R}^4$, $V = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} \right\}$.

   (a) (5 points) Three of the four vectors in V form an orthogonal set. Find these three vectors.

   Solution: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} \right\}$ is an orthogonal set.

   (b) (5 points) Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ denote the three vectors found in part (a) and let $\mathbf{u}$ denote the vector not used in (a). Write $\mathbf{u}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$. That is, find coefficients $a_1, a_2, a_3 \in \mathbb{R}$ so that $\mathbf{u} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3$.

   Solution: We have $\mathbf{u} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. We solve the linear system $\mathbf{u} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3$ to get $a_1 = 0, a_2 = \frac{1}{3}$, and $a_3 = \frac{1}{3}$.

   Remark: Notice that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ doesn't actually span $\mathbb{R}^4$, so it is not obvious that $\mathbf{u}$ is in the span. I basically told you it was in the problem, and if you solve the system you'll see that indeed it is a consistent system.

   I just wanted to point this out because if you used projections, you should really at least double check your answer to make sure you actually get the right answer. Indeed, if you use projections you get the same answer for a_1, a_2 and a_3, and you can check that $\frac{1}{3}\mathbf{v}_2 + \frac{1}{3}\mathbf{v}_3 = \mathbf{u}$.

2. For each of the following statements, decide whether it is true or false, and *justify your answers.*

(a) (3 points) True or false: If A and B are symmetric and $AB = BA$ (i.e. A and B commute) then AB is symmetric.

True. We have $(AB)^T = B^T A^T = BA$.

(b) (3 points) True or false: The set $W = \left\{ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ such that } x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \right\}$ is a subspace of $\mathbb{R}^3$.

False. W is not closed under scalar multiplication. For example, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in W$ but $\begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \notin W$.

(c) (3 points) True or false: Suppose A is an $n \times n$ singular (i.e. not invertible) matrix. Then zero is the only eigenvalue of A.

False. Consider for example the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. This is not invertible, and has eigenvalues 0 and 1.

(d) (3 points) True or false: If A is an $n \times n$ matrix with linearly independent columns, the rows are also linearly independent.

True. If the n columns are linearly independent, A has rank n. Therefore the dimension of the row space is n, and so the n rows are linearly independent. Note that is important that the number of rows and number of columns is the same here.

3. For each of the following statements, decide whether it is true or false. For this part, you do not need to justify your answers.

   (a) (2 points) True or false: If the linear system $A\mathbf{x} = \mathbf{y}$ has a unique solution $\mathbf{x}$, then $\mathbf{x}$ is the unique least squares solution to $A\mathbf{x} = \mathbf{y}$.

   True. Since the linear system has a solution, any least squares solution must be a solution. (since $\|A\mathbf{x} - \mathbf{y}\|$ has better be zero.) Since $\mathbf{x}$ is the only solution, it's the unique least squares solution.

   (b) (2 points) True or false: If A is an $n \times m$ matrix with linearly independent columns, then $A^T A = AA^T$.

   False. Consider for example $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. (You could even choose a matrix that isn't square and that would make it so $A^T A$ and AA^T have different dimensions.)

   (c) (2 points) True or false: If A is invertible, $\det(AB^T A^{-1}) = \det(B)$.

   True. $\det(AB^T A^{-1}) = \det(A)\det(B^T)\det(A^{-1}) = \det(B)$.
   (Remark: I was also assuming here that B was square, but if it weren't then these operations wouldn't even be defined.)

   (d) (2 points) True or false: If A is an $n \times m$ matrix that has rank m, then $A^T A$ is invertible.

   True. Since A has rank m, the columns are linearly independent. By a theorem in class, that means $A^T A$ is linearly independent.
   (The proof went something like this: First notice that $A^T A$ is square. Now, if $A^T A\mathbf{x} = \mathbf{0}$, then $\mathbf{x}^T A^T A\mathbf{x} = 0$, but this is the same as $\|A\mathbf{x}\|^2$. Therefore $A\mathbf{x} = \mathbf{0}$, and since the columns of A are linearly independent we conclude $\mathbf{x} = \mathbf{0}$. Therefore $A^T A\mathbf{x} = \mathbf{0}$ has only the trivial solution, and so it is invertible by the big theorem.)

   (e) (2 points) True or false: The function $f(x) = x + 1$ is a linear transformation from $\mathbb{R}$ to $\mathbb{R}$.

   False. For example, take $x = 0$ and $r = 0$. Then $rf(x) = 0$, but $f(rx) = 1$.

   (f) (2 points) True or false: If $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation, then T is one-to-one if and only if $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

   True. This was a theorem in the book. You can prove it as follows. It is immediate that if T is one-to-one then $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution. In the other direction, you show that if $T(\mathbf{u}) = T(\mathbf{v})$ then since T is linear we get $T(\mathbf{u} - \mathbf{v}) = \mathbf{0}$, and so $\mathbf{u} = \mathbf{v}$. Therefore T is one-to-one.

4. Suppose A is the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$. To help with calculations, the reduced echelon form of A is $B = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

(a) (5 points) Give a basis for the null space of A, null(A).

Solution: $Ax = 0$ if and only if $Bx = 0$, which we can see is true if and only if $x_1 = x_3$ and $x_2 = -2x_3$. This is true if and only if $x = \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. So a basis for the null space is $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$.

(b) (5 points) Give a basis for the orthogonal complement of null(A), that is, give a basis for null$(A)^\perp$.

The orthogonal complement is the set of all things orthogonal to everything in null(A). In this case, it's the set of all vectors orthogonal to $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$, and so one way to do this problem is to just solve the linear system $x_1 - 2x_2 + x_3 = 0$. null$(A)^\perp$ is exactly the set of solutions.
Another way to do this problem is to use that the row space is the orthogonal complement of the null space. But if you didn't know that, the first way is fine.

There is more than one possible basis, but one basis for the orthogonal complement is $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Remark: You should at least check that the basis elements are orthogonal to $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

5. Suppose $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$.

(a) (5 points) Compute $\det(A^7B^{-1}A)$.

Solution: $\det(A^7B^{-1}A) = \det(A)^7 \cdot 1/\det(B) \cdot \det(A) = (-1)^7 \cdot \frac{1}{2} \cdot (-1) = \frac{1}{2}$.
Remark: I think one person did this successfully the hard way. It might be possible, but it really isn't worth it. Use the tools we learned.

(b) (5 points) Compute the matrix ABA^{-1}.

$A^{-1} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$. We then do the matrix multiplication to get

$$ABA^{-1} = \begin{bmatrix} 2 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}.$$

6. (6 points) Choose **ONE** of the following to prove or disprove. Make it very clear which you are choosing, and whether you are proving or disproving it.
(Recall that for any sets A and B, $A \subset B$ means that all elements of A are contained in B, i.e. $\mathbf{x} \in A$ implies $\mathbf{x} \in B$).

   (a) Prove or disprove: If S is subspace of $\mathbb{R}^n$, and T is a subset of $\mathbb{R}^n$ with $S \subset T$, then T is a subspace of $\mathbb{R}^n$.

   (b) Prove or disprove: If S is subspace of $\mathbb{R}^n$, and T is a subset of $\mathbb{R}^n$ with $T \subset S$, then T is a subspace of $\mathbb{R}^n$.

Solution: These are both false, and there are a ton of counterexamples. To get full points on this problem, you just had to give one example that fails - say exactly what subspace S you're talking about, and what set T you're talking about.

7. Let T be a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$ and let

$$T\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \text{ and } T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

(a) (5 points) Compute $T\left(\begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}\right)$.

The idea here is to write $\begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. We can then use linearity of T to show that

$$T\left(\begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}.$$

(b) (5 points) Find a vector $\mathbf{w} \in \mathbb{R}^3$ such that $T(\mathbf{w}) = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$.

Again, we don't have to actually find out what T is to do this problem, we can just use what we know about it. We can write $\begin{bmatrix} 8 \\ 0 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 4\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Then we can let $\mathbf{w} = 4\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + 4\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 16 \end{bmatrix}$.

Remark: This is actually the only choice that will work for *any* T that satisfies the above properties. Some of you found a particular A that satisfied the above and then solved for $\mathbf{w}$, but if you had chosen a different A your choice wouldn't work.

8. We say a matrix A is **idempotent** if $A^2 = A$.

(a) (5 points) Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be an orthonormal basis for $\mathbb{R}^3$. Thinking of $\mathbf{u}_1$ as a 3×1 matrix, show that the 3×3 matrix given by $\mathbf{u}_1\mathbf{u}_1^T$ is idempotent.
(Remark: your arguments must be valid for *any* choice of orthonormal basis $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.)

Solution: $(\mathbf{u}_1\mathbf{u}_1^T)^2 = (\mathbf{u}_1\mathbf{u}_1^T)(\mathbf{u}_1\mathbf{u}_1^T) = \mathbf{u}_1(\mathbf{u}_1^T\mathbf{u}_1)\mathbf{u}_1^T = \mathbf{u}_1(\|\mathbf{u}_1\|^2)\mathbf{u}_1^T = \mathbf{u}_1(1)\mathbf{u}_1^T = \mathbf{u}_1\mathbf{u}_1^T$.

Remark: It is not true that $\mathbf{u}_1\mathbf{u}_1^T$ will be the identity matrix. In fact, this will always be a rank one matrix. (And has eigenvalue zero, as you will see in part (b).) It is also not true that $\mathbf{u}_1$ has to be $\mathbf{e}_i$ for some i. In fact, $\mathbf{u}_1$ can be any unit vector in $\mathbb{R}^3$. (There are infinitely many choices).

(b) (5 points) What are the eigenvectors and associated eigenvalues of $\mathbf{u}_1\mathbf{u}_1^T$? (Hint: what could the eigenvectors possibly be?)

The eigenvectors are $\mathbf{u}_1$ (with eigenvalue 1) and $\mathbf{u}_2$ and $\mathbf{u}_3$ (with eigenvalue 0). You can see this just by writing $\mathbf{u}_1\mathbf{u}_1^T\mathbf{u}_i = \mathbf{u}_1(\mathbf{u}_1 \cdot \mathbf{u}_i)$ and noticing that the dot product is either 1 if $i = 1$ or 0 otherwise.