Chapter 3: Linear Transformations

Chapter 3 is all about linear transformations, which are a particular type of function from $\mathbb{R}^m$ to $\mathbb{R}^n$. It is important that you know the definition of a linear transformation, along with many other things. The following are meant to be helpful, but do not constitute a comprehensive review. I recommend looking through the text - the midterm will cover 3.1-3.3 and 4.1-4.3, with the exception of the parts on partitioned matrices. Also there are tons of things left incomplete or unsaid in this write-up. Question everything, and try answering your questions.

Fact 1: Any linear transformation can be expressed as multiplication by a matrix.
That is, if $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation, then there is an $n \times m$ matrix A so that $T(\mathbf{x}) = A\mathbf{x}$. (Note: if you do not feel comfortable with matrix multiplication, review it! Do lots of examples!)
Why is it true?: We proved this by figuring out exactly what the columns of A should be. We have $A = [T(\mathbf{e}_1)T(\mathbf{e}_2)...T(\mathbf{e}_m)]$. (Again, if you are not comfortable with this, review it. See notes or the proof of theorem 3.8 in the book.)
How is it helpful?: If you have a linear transformation and want to figure out the matrix so that $T(x) = Ax$, you can always just figure out what $T(\mathbf{e}_1)$, ... $T(\mathbf{e}_m)$ are. This is also helpful because it is sometimes easier to think about matrices than functions. For example, it is useful to think about the range of T as the span of the columns of the matrix A, and it is helpful for knowing when T is one-to-one, onto, etc.

Fact 2: If S, T are linear transformations so that their composition is defined, then the composition is also a linear transformation. That is, if $R(\mathbf{x}) = T(S(\mathbf{x}))$, where T, S are linear transformations, then R is a linear transformation.
Why is it true?: This is written out in the in-class participation quiz. Solutions are on the website.
How is it helpful?: This means that there is a matrix that represents this composition of functions. We defined matrix multiplication to respect function composition, so that it works out that $(AB)x = A(Bx)$. (again, assuming the dimensions are appropriate).
If you don't feel comfortable with matrix multiplication, review it. If you like the way of calculating the matrix product via the dot product of rows of A and columns of B, that's great, but *make sure you also know the definition.*
In this chapter, we also talked about what it means for a function (and hence, for a linear transformation) to be one-to-one, onto and invertible. We know a linear transformation is invertible if and only if it is one-to-one and onto.

Fact 3: A linear transformation T is one-to-one if and only if $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution. **Why is it true?** See the proof of theorem 3.5.
How is it helpful? This is useful for thinking about how if $T(\mathbf{x}) = A\mathbf{x}$, then the columns of A being linearly independent is related to T being one-to-one and the kernel of T being $\{\mathbf{0}\}$, etc.

Fact 4: If T is an invertible linear transformation, then T^{-1} is also a linear transformation.
Why is it true?: This was proved in the notes (and is in the book) using the definition of the inverse and the definition of linear transformations. You are not required to know this proof, but it is kind of interesting.

How is it helpful?: Suppose $T(x) = Ax$ is an invertible linear transformation. Since the inverse function is always a linear transformation, we know that there will be a matrix representing the inverse linear transformation. This matrix ends up being precisely A^{-1} (which is defined to be the matrix so that $AA^{-1} = I$.)

You should know the definitions of matrix multiplication, transpose, symmetric matrices, inverse of a matrix, upper/lower triangular matrices, diagonal matrices, and powers of a matrix. You should also know properties of matrix algebra, as written in theorems 3.11, 3.13, 3.15, 3.16 and 3.17. (In fact, 3.17 holds for products as well as powers.) You should know what one-to-one and onto are, and what the range and kernel of a linear transformation are, and what the null space of a matrix is. You should know how all of these things are connected (and how they relate to spanning/ linear independence/ etc.) You should also know the statements and the proofs of all four parts of theorem 3.23. You should know how to find the inverse matrix. You should know how inverses relate to the "big theorem" (see theorem 3.24 - more details about the big theorem written out separately). You do not need to know the sections on "partitioned matrices". You should know the vocabulary well and be able to use the words appropriately. Remember that a lot of what we learn in this class is a tool to help you solve problems and think about things in a more clear way. This very much includes the vocabulary - we learn these words to be able to communicate better.