Summary of big theorem

As I've mentioned in class, the big theorem only applies when $m = n$. But regardless of what m and n are, there are still a lot of equivalences that are important to know. Please also keep in mind that equivalent here means that the statements are all true or all false - it doesn't mean they mean the same thing. It is important that you know what all of the parts mean.

Theorem 1. *Suppose* $T : \mathbb{R}^m \to \mathbb{R}^n$ *a linear transformation and* A *is an* $n \times m$ *matrix with* $T(\mathbf{x}) = A\mathbf{x}$*, and* $A = [\mathbf{a}_1 ... \mathbf{a}_m]$*. If* $m > n$*, none of the following hold. If* $m \leq n$*, the following are equivalent. (i.e. they are all true or all false.)*

1. *The columns of* A *are linearly independent (i.e.* $\{\mathbf{a}_1, ..., \mathbf{a}_m\}$ *is a linearly independent set.)*
2. T *is one-to-one.*
3. $A\mathbf{x} = \mathbf{b}$ *has at most one solution for all* $\mathbf{b} \in \mathbb{R}^n$*.*
4. $\ker(T) = \{\mathbf{0}\}$*.*
5. $\text{rank}(A) = m$*.*

Theorem 2. *Suppose* $T : \mathbb{R}^m \to \mathbb{R}^n$ *a linear transformation and* A *is an* $n \times m$ *matrix with* $T(\mathbf{x}) = A\mathbf{x}$*, and* $A = [\mathbf{a}_1 ... \mathbf{a}_m]$*. If* $m < n$*, none of the following hold. If* $m \geq n$*, the following are equivalent. (i.e. they are all true or all false.)*

1. $\{\mathbf{a}_1, ..., \mathbf{a}_m\}$ *spans* $\mathbb{R}^n$ *(i.e.* $\text{span}\{\mathbf{a}_1, ..., \mathbf{a}_m\} = \mathbb{R}^n$*.)*
2. T *is onto.*
3. $A\mathbf{x} = \mathbf{b}$ *has at least one solution for all* $\mathbf{b} \in \mathbb{R}^n$*.*
4. $\text{col}(A) = \mathbb{R}^n$*.*
5. $\text{rank}(A) = n$*.*

Theorem 3 ("Big Theorem")**.** *Now suppose we are in the situation of theorems 1 and 2, with* $m = n$*. Then all of the conditions of theorems 1 and 2, and all of the following, are equivalent. (either all 15 statements are true or all are false)*

1. A *is an invertible matrix.*
2. $\{\mathbf{a}_1, ..., \mathbf{a}_m\}$ *is a basis for* $\mathbb{R}^n$*.*
3. $A\mathbf{x} = \mathbf{b}$ *has exactly one solution for all* $\mathbf{b} \in \mathbb{R}^n$*.*
4. $\det(A) \neq 0$*. (won't cover until chapter 5)*
5. $\lambda = 0$ *is not an eigenvalue of* A*. (won't cover until chapter 6)*