Math 308O **Name (Print):** ____________________
Winter 2015
Midterm 2
February 25, 2015
Time Limit: 50 Minutes Instructor ____________________

This exam contains 5 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use a scientific calculator and a sheet of notes, 8.5×11 and handwritten by you, but no other devices, books, or notes are permitted.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. I can also provide paper to attach to your exam if needed.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

1. Let $A = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 4 & 4 & 7 & 2 \\ 1 & 3 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$.

   (a) (5 points) Compute the rank of A.

   Solution: We can reduce the matrix to echelon form. Since the echelon form has four nonzero rows, the matrix has rank four.

   (b) (3 points) Compute the nullity of A.

   Solution: rank+nullity=m=4, so we know that nullity= 0.

   (c) (2 points) Is A invertible? Explain.

   Solution: Yes. The big theorem tells us that A is invertible if and only if the rank of the matrix is equal to $n = 4$.

2. Read each of the following statements carefully, and decide whether it is true or false. You are not required to justify your answers, but I recommend justifying them to yourself.

(a) (2 points) Suppose $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation. If $m < n$, then T is one-to-one.

False. We don't know whether or not T will be one-to-one. For example, we could have $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(\mathbf{x}) = \mathbf{0}$. This would not be one-to-one.

(b) (2 points) Suppose A is an $(n \times n)$ matrix with linearly independent columns, and define $T(\mathbf{x}) = \mathrm{A}\mathbf{x}$. Then T is onto.

True. A is a square matrix with linearly independent columns, and so by the big theorem T is invertible.

(c) (2 points) Suppose A and B are nonsingular $(n \times n)$ matrices. Then $(AB)^2 = A^2B^2$.

False. It can happen that A and B don't commute, which will usually give us that these are not equal. (Recall that $(AB)^2 = ABAB$, while $A^2B^2 = AABB$). Certainly they could be equal for some A, B, but they are not always equal.

(d) (2 points) If A is an $(n \times n)$ matrix with linearly independent column vectors, the row vectors of A are also linearly independent.

True. The columns being linearly independent means the dimension of the column space is n, and so the dimension of the row space is also n. Therefore the n rows must be linearly independent. (Question: What about if A is not square?)

(e) (2 points) True or false: If A is an $(n \times n)$ matrix, then $A + A^T$ is symmetric.

True. We can use properties of transposes to show that $(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$. The matrix is equal to its own transpose, which is what it means to be symmetric.

3. Suppose A is a nonsingular (3×3) matrix with $A^{-1} = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 3 & 1 \\ 3 & -8 & 4 \end{bmatrix}$.

(a) (5 points) If $A^T x = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, find x. Note that you do not need to compute A to do this calculation.

Solution: $(A^T)^{-1} = (A^{-1})^T = \begin{bmatrix} 0 & -1 & 3 \\ 1 & 3 & -8 \\ 2 & 1 & 4 \end{bmatrix}$.

Therefore $x = \begin{bmatrix} 0 & -1 & 3 \\ 1 & 3 & -8 \\ 2 & 1 & 4 \end{bmatrix} \times \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ 5 \end{bmatrix}$.

(b) (5 points) Suppose B is another nonsingular matrix with $B^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Compute the matrix $(AB)^{-1}$.

Solution: $(AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 2 \\ -1 & 3 & 1 \\ 3 & -8 & 4 \end{bmatrix} = \begin{bmatrix} 2 & -5 & 5 \\ 0 & 2 & 9 \\ 3 & -8 & 4 \end{bmatrix}$.

4. (10 points) Choose **one** of the following statements to prove. Please note you will not get additional credit for doing some of each. Make it **VERY CLEAR** which proof you want me to grade.

   (a) Suppose that A is an $(n \times m)$ matrix. Prove that the null space of A is a subspace of $\mathbb{R}^m$.

   Solution: The null space of A is the set of vectors $\mathbf{x}$ in $\mathbb{R}^m$ satisfying $A\mathbf{x} = \mathbf{0}$. We want to show this is a subspace of $\mathbb{R}^m$. We already know it is a subset, so we just have to check if the three conditions are satisfied.
   First, $\mathbf{0}$ is in null(A) because $A\mathbf{0} = \mathbf{0}$.
   To check the second condition, suppose that $\mathbf{u}$ and $\mathbf{v}$ are elements of null(A). Then $A\mathbf{u} = \mathbf{0}$ and $A\mathbf{v} = \mathbf{0}$. Therefore $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}$, and hence $\mathbf{u} + \mathbf{v}$ is in null(A). Since $\mathbf{u}$ and $\mathbf{v}$ were arbitrary elements of null(A), the second condition is satisfied.
   To check the third condition, suppose that $\mathbf{u}$ is in null(A) and r is a real number. Then $A(r\mathbf{u}) = r(A\mathbf{u}) = r\mathbf{0} = \mathbf{0}$, and so $r\mathbf{u}$ is in null(A). Since r and $\mathbf{u}$ were arbitrary, the third condition is satisfied.
   Therefore we can conclude that the null space is a subspace of $\mathbb{R}^m$.

   (b) Suppose that $T : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear transformation. Prove that T^{-1} is also a linear transformation.

   Solution: Since T is invertible, it is one-to-one and onto. We want to show that T^{-1} is a linear transformation. To prove this, we need to verify the two conditions to be a linear transformation. To check the first condition, suppose that $\mathbf{y_1}$ and $\mathbf{y_2}$ are in $\mathbb{R}^n$. Since T is onto, there are some $\mathbf{x_1}$ and $\mathbf{x_2}$ so that $T(\mathbf{x_1}) = \mathbf{y_1}$ and $T(\mathbf{x_2}) = \mathbf{y_2}$.
   We then have $T^{-1}(\mathbf{y_1} + \mathbf{y_2}) = \mathrm{T}^{-1}(\mathrm{T}(\mathbf{x_1}) + \mathrm{T}(\mathbf{x_2})) = \mathrm{T}^{-1}(\mathrm{T}(\mathbf{x_1} + \mathbf{x_2}))$ since T is a linear transformation. But we also know that for any $\mathbf{x}$, $T^{-1}(T(\mathbf{x})) = \mathbf{x}$, and so the above equation simplifies to $T^{-1}(\mathbf{y_1} + \mathbf{y_2}) = \mathbf{x_1} + \mathbf{x_2} = \mathrm{T}^{-1}(\mathbf{y_1}) + \mathrm{T}^{-1}(\mathbf{y_2})$. Therefore the first condition holds.
   To check the second condition, suppose $\mathbf{y}$ is in $\mathbb{R}^n$ and r is some real number. Then since T is onto, there is some $\mathbf{x}$ so that $T(\mathbf{x}) = \mathbf{y}$. Then $T^{-1}(r\mathbf{y}) = \mathrm{T}^{-1}(\mathrm{rT}(\mathbf{x})) = \mathrm{T}^{-1}(\mathrm{T}(\mathrm{r}\mathbf{x})) = \mathrm{r}\mathbf{x} = \mathrm{rT}^{-1}(\mathbf{y})$.
   Therefore the second condition holds. Since both conditions hold, T^{-1} is a linear transformation.