1. Suppose we know that a polynomial takes the form $f(x) = ax^2 + bx + c$, and goes through the points $(1, 1), (2, 1)$, and $(-1, -5)$. The goal of this problem will be to find a, b and c.

   (a) (3 points) Set up a system of linear equations and find the corresponding augmented matrix.

$$a + b + c = 1$$
$$4a + 2b + c = 1$$
$$a - b + c = -5$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 4 & 2 & 1 & 1 \\ 1 & -1 & 1 & -5 \end{bmatrix}$$

   (b) (4 points) Put the augmented matrix into echelon form.

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -3 & -3 \\ 0 & -2 & 0 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -3 & -3 \\ 0 & 0 & 3 & -3 \end{bmatrix}$$

   (c) (3 points) Solve for a, b and c. Please check your work.

Using back substitution

$$c = -1$$

$$-2b - 3c = -3$$
$$\Rightarrow b = -\frac{1}{2}(-3 + 3c) = -\frac{1}{2}(-6) = 3$$

$$a + b + c = 1$$
$$\Rightarrow a = 1 - b - c$$
$$= 1 - (3) - (-1) = -1$$

$$f(x) = -x^2 + 3x - 1$$

$$f(1) = -1 + 3 - 1 = 1 \checkmark$$
$$f(2) = -4 + 6 - 1 = 1 \checkmark$$
$$f(-1) = -1 - 3 - 1 = -5 \checkmark$$

2. Read each of the following statements carefully, and decide whether it is true or false. You are not required to justify your answers, but I recommend justifying them to yourself.

(a) (2 points) A homogeneous linear system with n variables and n equations will have exactly one solution.

False. It might have infinitely-many.

For example, $x_1 + x_2 = 0$

$2x_1 + 2x_2 = 0$.

(b) (2 points) Suppose $\mathbf{x_1}$ and $\mathbf{x_2}$ are vectors in $\mathbb{R}^m$, $\mathbf{b}$ is a vector in $\mathbb{R}^n$, and A is an $n \times m$ matrix. If $\mathbf{x_1}$ and $\mathbf{x_2}$ are both solutions to the equation $A\mathbf{x} = \mathbf{b}$, then so is the vector $\mathbf{x_1} + \mathbf{x_2}$.

False. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{x}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Then $A\vec{x}_1 = A\vec{x}_2 = b$, but $A(\vec{x}_1 + \vec{x}_2) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \neq \vec{b}$.

(c) (2 points) The set $\left\{ \begin{bmatrix} 2 \\ 3 \\ 7 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 15 \\ 0 \\ 0 \end{bmatrix} \right\}$ is linearly dependent.

True. One way to see this is to see that these are four vectors that do not span $\mathbb{R}^4$. (In particular $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ is not in the span). Therefore by the "big theorem" they are lin. dep.

(d) (2 points) If the columns of A are linearly dependent, then the equation $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

False. It might have no solutions.

For example, $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(e) (2 points) If the set $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ is linearly independent in $\mathbb{R}^3$, then the set $\{\mathbf{0}, \mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ spans $\mathbb{R}^3$.

True. Since $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is a linearly independent set of 3 vectors in $\mathbb{R}^3$ the big theorem tells us it spans $\mathbb{R}^3$. Therefore so does the set $\{\vec{0}, \vec{u}_1, \vec{u}_2, \vec{u}_3\}$.

3. (a) (3 points) Find vectors $\mathbf{u_1}, \mathbf{u_2}$ and $\mathbf{u_3}$ in $\mathbb{R}^3$ so that the sets $\{\mathbf{u_1}, \mathbf{u_2}\}$, $\{\mathbf{u_2}, \mathbf{u_3}\}$ and $\{\mathbf{u_1}, \mathbf{u_3}\}$ are linearly independent, but the set $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ is linearly dependent.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Any two of them are independent, because they aren't scalar multiples of each other.

But ~~any three~~ all three are dependent because they don't span and there are three of them. (Same reasoning as #2c)

(b) (3 points) Find five distinct nonzero vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}, \mathbf{v_5}$ in $\mathbb{R}^3$ so that $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}, \mathbf{v_5}\}$ does not span $\mathbb{R}^3$.

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 7 \\ 0 \end{bmatrix}$$

There are a <u>ton</u> of possibilities for this. If you have an idea you want to run past me feel free to ask.

(c) (4 points) Find a value for k so that the set $\left\{ \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ k \end{bmatrix} \right\}$ is linearly dependent.

$$\begin{bmatrix} 2 & 4 & 2 & 0 \\ 3 & 2 & -1 & 0 \\ 5 & 2 & k & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 2 & -1 & 0 \\ 5 & 2 & k & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -4 & -4 & 0 \\ 0 & -8 & k-5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -8 & k-5 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & k+3 & 0 \end{bmatrix}$$

This is linearly dependent when this has ∞ many solns, which happens precisely when $\boxed{k = -3}$

4. (10 points) Give a geometric interpretation (including the exact equation) for

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix} \right\}.$$

~~dependent ⇒ 3 points~~

~~plane ⇒ 5 points~~

~~equation + plane ⇒ 10 points~~

~~Incorrectly conclude $\mathbb{R}^3$ ⇒ 5 points (with good reason)~~

The vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is in the span of these vectors when the following system is consistent.

$$\begin{bmatrix} 1 & 1 & -1 & x \\ 1 & 2 & 0 & y \\ 1 & 0 & -2 & z \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 & x \\ 0 & 1 & 1 & y-x \\ 0 & -1 & -1 & z-x \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & x \\ 0 & 1 & 1 & y-x \\ 0 & 0 & 0 & -2x+y+z \end{bmatrix}$$

Thus $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is in the span if and only if $-2x + y + z = 0$.

This is the equation of a plane in $\mathbb{R}^3$.