Lecture #11, Wednesday, May 5, 1999

Handouts:
Agenda: WS #17 (continued); WS #18:
Announcements — Section A

- **NOTEBOOKS**: Some notebooks are still outside of my office (337-C Padelford). Please pick yours up today.

There will be a second check of notebooks in about two weeks. I expect to grade two more problems at random, each worth 3 points. I will also award up to 3 points based on the level of completeness and general readability of the portion of your notebook between WS #14 and whatever WS is the last one before the day I collect them.
Discussion of $D$

$D$ depends on both $t$ and $m$. It is a function of TWO variables:

$$D = f(t, m) = -2t^2 + 48t - 4(t - m)^2$$

We have special names and symbols for the derivatives above:

Suppose that we fix $m = m^*$, a constant. Then $f(t, m^*)$ is a function of only the single variable $t$ and we can compute its derivative. We call this derivative the Partial Derivative of $f(t, m^*)$ with respect to $t$:

$$\frac{\partial f(t, m^*)}{\partial t} = (-2)(2)t + 48 - (4)(2)(t - m^*)$$

$$= -12t + 8m^* + 48$$
Similarly if we fix $t = t^*$ a constant, then $f(t^*, m)$ is a function of the variable $m$, and we call the derivative with respect to $m$ the **Partial Derivative** of $f(t^*, m)$ with respect to $m$:

$$\frac{\partial f(t^*, m)}{\partial m} = 0 + 0 - (4)(2)(t^* - m)(-1)$$

$$= 8(t^* - m)$$
The Zero Derivative Principle:

Suppose that $D = f(t, m)$ is greatest at $(t, m) = (t^*, m^*)$. Then the following equations hold:

\[
\begin{align*}
\cdot \frac{\partial f(t^*, m^*)}{\partial t} &= 0 \\
\cdot \frac{\partial f(t^*, m^*)}{\partial m} &= 0
\end{align*}
\]

(Conclusion) The first step to finding the values of $(t, m)$ at which $D = f(t, m)$ achieves is Maximum (or Minimum) is to solve the system of equations

\[
\frac{\partial f(t, m)}{\partial t} = 0 \quad \frac{\partial f(t, m)}{\partial m} = 0
\]
Recall that:

\[ D = f(t, m) = -2t^2 + 48t - 4(t - m)^2 \]

So

\[ \frac{\partial f(t, m)}{\partial t} = -12t + 8m + 48 \]
\[ \frac{\partial f(t, m)}{\partial m} = 8t - 8m \]

We have to solve the system of equations

\[ \frac{\partial f(t, m)}{\partial t} = -12t + 8m + 48 = 0 \]
\[ \frac{\partial f(t, m)}{\partial m} = 8t - 8m = 0 \]
To solve the system of equations

\[ \frac{\partial f(t, m)}{\partial t} = -12t + 8m + 48 = 0 \]
\[ \frac{\partial f(t, m)}{\partial m} = 8t - 8m = 0 \]

proceed as follows:

• Solve the second equation for \( m \) to get

\[ m = t \]

• Plug this into the first equation to get

\[ -12t + 8t + 48 = 0 \]

\[ -4t + 48 = 0. \]

• Solve for \( t \) to get \( t = 48/4 = 12 \).

• From the first step, \( m = t = 12 \)

• Hence(?), the max value of \( D \) occurs at:

\( (t, m) = (12, 12) \)
### Comparison of optimization problems

<table>
<thead>
<tr>
<th></th>
<th>( g(x) )</th>
<th>( k(t, m) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of variables</td>
<td>1 variable</td>
<td>2 variables</td>
</tr>
<tr>
<td>Local optima?</td>
<td>local optima?</td>
<td>local optima?</td>
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<tr>
<td>Solve ( g'(x) = 0 ) for ( x )</td>
<td>Solve the system</td>
<td></td>
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<tr>
<td></td>
<td>( \frac{\partial k(t, m)}{\partial t} = 0 )</td>
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<td>( \frac{\partial k(t, m)}{\partial m} = 0 )</td>
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<tr>
<td>Solutions are candidates</td>
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<tr>
<td>Screening candidates</td>
<td>Screen candidates</td>
<td></td>
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<tr>
<td>2nd derivative test</td>
<td>more complicated.</td>
<td></td>
</tr>
<tr>
<td>( g''(x) &gt; 0 ) local min</td>
<td>( g''(x) &lt; 0 ) local max</td>
<td></td>
</tr>
</tbody>
</table>
Second Derivatives

\[
\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right)
\]

means: first take the partial derivative of \( z \) with respect to \( y \), and then take the partial derivative of the result with respect to \( x \).

Similarly,

\[
\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial y} \right)
\]

means: first take the partial derivative with respect to \( y \) and then take the partial derivative of the result with respect to \( y \) again.

\[
\frac{\partial^2 z}{\partial y \partial x} \text{ and } \frac{\partial^2 z}{\partial x^2}
\]

have similar meanings.
Example:

Compute all of these derivatives for the function

\[ f(x, y) = \frac{x^3}{3} - 2xy + y^2 - 3x \]

Notice:

\[ \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}. \]

This will always happen with the functions you run into.
Example:

Find all second partial derivatives of the function

\[ z = f(x, y) = \frac{x^3}{3} - 2xy + y^2 - 3x \]

Example: Find all second partial derivatives of the function

\[ z = h(u, v) = u^4 + uv^2 + v^3 \]

NOTE:

\[ \frac{\partial^2 f(x, y)}{\partial x \partial y} = \frac{\partial^2 f(x, y)}{\partial y \partial x} \quad \text{and} \quad \frac{\partial^2 h(u, v)}{\partial u \partial v} = \frac{\partial^2 h(u, v)}{\partial v \partial u} \]

This happens for all of the functions you will see in this course!
Second Derivative Test

Suppose you have a function \( z = F(x, y) \) and then point \((a, b)\) gives zero partial derivatives, i.e. \( \frac{\partial z}{\partial x}(a, b) = 0 \) and \( \frac{\partial z}{\partial y}(a, b) = 0 \). Then

Calculate the number
\[
D = \frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} - \left( \frac{\partial^2 z}{\partial x \partial y} \right)^2 \at \text{the point} (a, b).
\]

(i) If \( D = 0 \), inconclusive (you can not tell anything from the test).

(ii) If \( D < 0 \), then you have a “saddle point” (you do not have a maximum or a minimum).

(iii) If \( D > 0 \), then:
- you have a maximum if \( \frac{\partial^2 z}{\partial x^2}(a, b) < 0 \)
- you have a minimum if \( \frac{\partial^2 z}{\partial x^2}(a, b) > 0 \).
Second Derivative Test

This can be summarized by the following chart:

<table>
<thead>
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<th>$D &gt; 0$</th>
<th>$D &lt; 0$</th>
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<tbody>
<tr>
<td>$rac{\partial^2 z}{\partial x^2}$ &gt; 0</td>
<td>min</td>
<td>saddle point</td>
</tr>
<tr>
<td>$rac{\partial^2 z}{\partial x^2}$ &lt; 0</td>
<td>max</td>
<td>saddle point</td>
</tr>
</tbody>
</table>
Visualizing function of 2-variables

NONE OF THE REST of this covered in Lecture

\[ z = f(x, y) = 2x^2 + 2xy + 5y^2 - 18x - 18y + 47. \]

As a Table of values:
Slide 15
As a “Contour Map”:

Connect points where \( f(x, y) \) has same value by a curve:

Small ellipse is \( f(x, y) = 4 \), then \( f(x, y) = 7 \), and so on ...
A Contour is a slice of part of a Surface:

Assemble the contours into a 3-D shape.
As a collection of curves given by slices:

Fix $x = x^*$ and let $y$-vary: $z = f(x^*, y)$
Fix $y = y^*$ and let $x$-vary: $z = f(x, y^*)$
The Curve given by the slice \( x = 2 \)

\[ z = f(2, y) \]