Math 380 **HOMEWORK #5** **due Tue, November 1, 2017**

Written assignment (4 problems).

W1. Show that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ can never be an integer.

W2. Suppose $p(x) = \sum_{i=0}^{r} a_i x^i$ is a polynomial with all coefficients a_i integers. Assume that a_0, a_r, and $p(1)$ are all odd. Show that p cannot have rational roots.

W3. We call an integer *square-free* if it is the product of single powers of primes (i.e., no square power of a prime divides it). For example, 6 is square-free, while 12 is not.

Given $n \in \mathbb{N}$, show that one can find a large enough integer A such that *none* of the numbers A, $A+1$, $A+2$, ..., $A+n-1$ is square-free.

W4. Prove that $a_{m,n} = \frac{(2m)!(2n)!}{m!n!(m+n)!}$ is an integer for any two non-negative integers m, n.

EC1. Show that $\sum_{k=0}^{n} \binom{2n+1}{2k+1} 2^{3k}$ is not divisible by 5 for any integer $n \geq 0$.

EC2. Show that $n^2/2 < \sigma(n)\phi(n) < n^2$.

Presentation assignment (5 problems).

P1. Let f be a function defined on positive integers such that $f(1) = 1$, $f(2n) = f(n)$, and $f(2n+1) = f(n) + 1$. Find an expression for f.

P2. A polynomial with integer coefficients is called *primitive* if its coefficients are relatively prime.

1. Show that a product of two primitive polynomials is primitive.
2. Prove **Gauss' Lemma**: If a polynomial with integer coefficients can be factored as a product of two non-constant polynomials with rational coefficients then it can also be factored as a product of two no-constant polynomials with integer coefficients.

P3. A positive integer n is called *perfect* if $\sigma(n) = 2n$.

1. Show that if $2^k - 1$ is a prime then $2^{k-1}(2^k - 1)$ is perfect.
2. Prove the partial converse: every even perfect number must be of the form $2^{k-1}(2^k - 1)$, where $2^k - 1$ is a prime.

P4. Let p be an odd prime and let $P(x)$ be a polynomial of degree at most $p - 2$.

1. Show that if $P(x)$ has integer coefficients then

$$P(n) + P(n+1) + \cdots + P(n+p-1)$$

is divisible by p for any integer n.

2. Conversely, suppose that

$$P(n) + P(n+1) + \cdots + P(n+p-1)$$

is divisible by p for any integer n. Is it true that $P(x)$ must have integer coefficients?

P5. Find all integers n such that $x^4 - 2(n+2017)x^2 + (n-2017)^2$ factors into two polynomials with integer coefficients.