A Boundary Value Problem for the Einstein Constraint Equations

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1. Introduction

The N-body problem in general relativity concerns the dynamics of an isolated system of N black holes. One aspect of the problem, quite different from its classical counterpart, is the complexity of constructing appropriate initial data for the associated Cauchy problem. Initial data for the Cauchy problem in general relativity is a set (M, g, K) where (M, g) is a Riemannian manifold and K is a symmetric $(0, 2)$-tensor. The data (M, g, K) is not freely specifiable, but instead must satisfy a system of PDEs known as the Einstein constraint equations. Initial data for the N-body problem should evolve into a spacetime containing the desired number of black holes. This last requirement turns out to be difficult to satisfy since the boundary of a black hole is a global property of a spacetime which can only be detected after initial data has been evolved. There do exist structures, apparent horizons and trapped surfaces, that can be detected in initial data and indicate the development of a black hole in the spacetime. To create initial data for N black holes, we create initial data containing N apparent horizons. There is no assurance, however, that distinct apparent horizons will be contained in distinct black holes; we can only presume that if the horizons are well separated that this will indeed be the case.

Traditional methods of generating initial data containing trapped surfaces and apparent horizons do so indirectly by working with manifolds with nontrivial topology; these methods guarantee the existence of an apparent horizon somewhere in the data but typically do not dictate precisely where. A direct approach to the problem, first proposed for numerical study in [Th87], is to work with a manifold with boundary and specify that the boundary be an apparent horizon. Until recently, however, there has not been a rigorous construction of such initial data. This year saw the near simultaneous appearance of two preprints treating versions of this problem: [Ma03] constructing initial data with apparent horizon boundaries and [Da03] constructing initial data with trapped surface boundaries. In this paper we provide background material needed to motivate the boundary value problems studied in these papers, and we outline the proof from [Ma03]. Section 2 provides basic material from Lorentzian geometry and general relativity, especially the relationship between trapped surfaces and black holes. The exposition in this section, reflecting the physical nature of the material, is not meant to be mathematically rigorous. Section 3 then contains an outline of the construction from [Ma03]. The goal of this section is to describe the construction, state the principal theorems precisely, and indicate the method of proof. For details of the proofs, the reader is referred to [Ma03]; the theorem numbers in Section 3 agree with those of [Ma03] for easy cross referencing. Section 4 then contains avenues for further research.

2. Motivation for the Boundary Value Problem

2.1 Lorentzian Manifolds and Causal Structure

A **Lorentzian manifold** is a smooth topological manifold M^n equipped with a metric g having signature $(-,+,\cdots,+)$. The metric g naturally partitions each tangent space into three regions. We say a vector X is **timelike**, **spacelike** or **null** if $\mathsf{g}(X,X)$ is negative, positive or zero respectively, and we say a vector is **causal** if it is either timelike or null. A Lorentzian manifold is said to be **time orientable** if there exists a continuous globally defined timelike vector field F. Such a vector field is said to give the manifold a time orientation, and any other timelike vector X is said to be **future or past pointing** if $\mathsf{g}(F,X)$ is negative or positive respectively. Henceforth we assume all Lorentzian manifolds are time oriented.

A hypersurface M of M is timelike, spacelike, or null if its tangent space at each point has a normal vector that is spacelike, timelike, or null respectively. A curve is timelike, spacelike, or null if it has a timelike, spacelike, or null tangent vector at each point. A curve is **timelike future directed** if its tangent vector at each point is timelike and future pointing (with analogous properties defined similarly). We say a point x **chronologically precedes** y if there exists a piecewise future directed timelike curve from x to y, in which case we write $x \ll y$. Similarly, x **causally precedes** y if there exists a piecewise future directed causal curve from x to y, and we write $x \prec y$. Note that degenerate curves are never timelike, but are always causal. Hence $x \prec x$ always holds, but it is not true in general that $x \ll x$.

The **chronological future** of a point x is the set $I^+(x) = \{y : x \ll y\}$, and the **chronological past** of x is $I^-(x) = \{y : y \ll x\}$. The **causal future** $J^+(x)$ and past $J^-(x)$ are defined similarly replacing $\ll$ with $\prec$. These definitions extend in the obvious way to sets. For example, $I^+(S) = \cup_{x\in S} I^+(x)$. It follows trivially from the definitions that $I^+(S) \subset J^+(S)$. Moreover by working in geodesic normal coordinates, it is easy to see that $I^+(S)$ is always open. It is not true in general that $J^+(S)$ is closed, even when S is compact. For example, consider Minkowski space $\mathbb{M}^n$, i.e. $\mathbb{R}^n$ equipped with the metric $\mathrm{diag}(-1,1,\cdots,1)$. If $x \in J^-(0)$ in $\mathbb{M}^n$, then $J^+(x)$ is not closed in $\mathbb{M}^n - 0$. So $\overline{I}^+(S) \supset J^+(S) \supset I^+(S)$ with each containment strict in general. There is a special kind of Lorentzian manifold for which $J^+(K)$ is always closed for compact sets K. A **Cauchy surface** of M is a spacelike hypersurface having the property that every inextendible timelike curve in M intersects M once and only once. Not every Lorentzian manifold admits a Cauchy surface; those that do are called **globally hyperbolic**. In a globally hyperbolic spacetime, $J^+(K) = \overline{I}^+(K)$ for all compact sets K, and in particular $J^+(K)$ is closed.

2.2 The Einstein Equation and the Initial Value Problem

In general relativity, spacetime is modeled by a 4-dimensional Lorentzian manifold (M, g). The metric g is assumed to satisfy the Einstein equation

$$\mathrm{Ric}_{\mathsf{g}} - \frac{1}{2} R_{\mathsf{g}}\, \mathsf{g} = T \tag{1}$$

where T is a symmetric $(0, 2)$-tensor, called the stress-energy tensor, dictated by the non-gravitational matter fields. In the vacuum case we have $T = 0$, and taking the trace of (1) yields the vacuum Einstein equation

$$\mathrm{Ric}_{\mathsf{g}} = 0. \tag{2}$$

Suppose M is a globally hyperbolic Lorentzian manifold with spacelike Cauchy surface M satisfying the vacuum Einstein equation. Let g be the Riemannian metric on M induced by g, let n be the future pointing timelike unit normal to M, and let K denote the extrinsic curvature of M computed with respect to n, i.e. $K(X, Y) = -\langle \nabla_X n, Y\rangle$ where ∇ is the connection on M. The Gauss-Codacci equations permit the computation of $n \lrcorner (\mathrm{Ric}_{\mathsf{g}} - \frac{R}{2} g)$ in terms of g and K. Since $\mathrm{Ric}_{\mathsf{g}} = 0$ we obtain

$$R - |K|^2 + \mathrm{tr}\, K^2 = 0 \tag{3}$$

$$\mathrm{div}\, K - d\ \mathrm{tr}\, K = 0. \tag{4}$$

where all quantities of (3) and (4) involving a metric are computed with respect to g. Equations (3) and (4) are known as the Hamiltonian and momentum constraint equations respectively. Together they form the Einstein constraint equations.

The (vacuum) initial value problem of general relativity is the following. Given initial data (M, g, K), find a Lorentzian manifold (M, g) satisfying the vacuum Einstein equations and an embedding $\iota : M \mapsto \mathsf{M}$ such that $\iota(M)$ is a Cauchy surface for M and such that g induces g and K on $\iota(M)$. A spacetime M satisfying these properties is called a **Cauchy development** of (M, g, K) and is by definition a globally hyperbolic Lorentzian manifold. If it is also true that every Cauchy development of (M, g, K) can be isometrically embedded in M we say M is called the **maximal development** of (M, g, K) (one can show a maximal development is unique up to isomorphism). We have previously noted that (M, g, K) must satisfy the constraint equations in order to have a Cauchy development. The following theorem shows that this condition is sufficient for there to exist a maximal development.

Theorem 2.1. [FB52][CBG69] *Given smooth initial data (M, g, K) satisfying the constraint equations (3) and (4), there exists a smooth maximal globally hyperbolic Cauchy development of the initial data.*

A version of the above theorem also holds with respect to less regular initial data. More will be said later about low regularity solutions of the constraint equations.

2.3 Black Holes and Scri

Heuristically, a black hole is a region of space that cannot send signals to infinity. One way to make this notion precise is the machinery of conformal completions. A Lorentzian manifold with boundary $(\mathsf{M}', \mathsf{g}')$ is a conformal completion of (M, g) if there exists an embedding $\iota : \mathsf{M} \mapsto \mathsf{M}'$ such that

1. $\iota(\mathsf{M}) = \operatorname{int} \mathsf{M}'$.
2. There exists a differentiable function Ω smooth on $\operatorname{int} \mathsf{M}'$, such that $\mathsf{g}' = \Omega^2 \iota_* \mathsf{g}$ in $\operatorname{int} \mathsf{M}'$.
3. Ω vanishes on $\mathscr{I} = \partial \mathsf{M}'$ and $d\Omega$ is nowhere vanishing on $\mathscr{I}$.

One typically also makes additional hypotheses concerning the smoothness of $\mathscr{I}$ and on the differentiability of Ω at $\mathscr{I}$, but the correct choice of these is still an area of research and is not important to the subsequent exposition. Since Ω vanishes at the boundary, we think of the boundary $\mathscr{I}$ (called Scri) as the boundary at infinity. Note that if M' is a conformal completion of M and if $p \in \mathscr{I}$, then $\mathsf{M}' - \{p\}$ is also a conformal completion of M. Hence any useful definition of conformal completion should also include a completeness hypothesis, which we will return to below.

We first recall the standard conformal completion of Minkowski space. Consider the cylindrical manifold $E = \mathbb{R} \times S^3$ with the metric $\mathsf{g}' = -dT^2 + dS_3^2$, where dS_3^2 is the round metric on S^3 and T is the coordinate on the $\mathbb{R}$ factor. We then embed $\mathbb{M}^4$ into E as follows. Let t and r denote the usual time and spacial radius coordinates of Minkowski space, and let R denote the distance in S^3 from the north pole. We then set

$$
\begin{aligned}
T &= \arctan(t + r) + \arctan(t - r) \\
R &= \arctan(t + r) - \arctan(t - r)
\end{aligned}
$$

and identify surfaces of constant r in $\mathbb{M}^4$ with surfaces of constant R in S^3 in the natural way to obtain an embedding of $\mathbb{M}^4$ into the region of E given by

$$
\begin{aligned}
-\pi < T + R < \pi \\
-\pi < T - R < \pi
\end{aligned}
$$

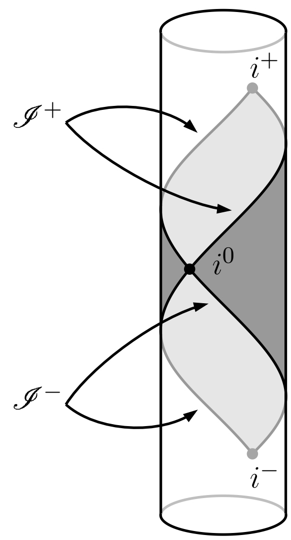

Figure 1: The Conformal Completion of $\mathbb{M}^4$as a subset of $S^3 \times \mathbb{R}$

Letting ι denote this embedding, then

$$\iota^* \mathsf{g}' = \Omega^2 \overline{\mathsf{g}}$$

where

$$\Omega^2 = \frac{4}{(1+(t+r)^2)(1+(t-r)^2)}.$$

We take M' to be the image of $\mathbb{M}$ under this embedding together with the null surfaces $\mathscr{I}^+$ and $\mathscr{I}^-$ indicated in Figure 1 (the points i^0, i^+, and i^- are not part of M'; the boundary is not regular at $i^\pm$ and $\mathsf{M}' \cup i^0$ is not a manifold with boundary).

For a general conformal completion, we define $\mathscr{I}^+ = \{x \in \mathscr{I} : I^- x \cap \mathsf{M} \neq \emptyset\}$, with an analogous definition for $\mathscr{I}^-$. Intuitively $\mathscr{I}^+$ is the portion of the boundary that can be reached by future oriented timelike curves starting in M. These definitions agree with the sets indicated in Figure 1.

Given a Lorentzian manifold M possessing a conformal completion M', we define the **black hole region** of M to be $\mathcal{B} = \mathsf{M} - J^-(\mathscr{I}^+)$, so that the black hole region is the part of M that cannot be seen from $\mathscr{I}^+$. However, the black hole region depends on the choice of conformal completion. To compensate for this, we say M is **asymptotically Minkowskian at future null infinity** if it has a conformal completion M' and moreover

4. $\operatorname{Hess}\Omega = 0$ on $\mathscr{I}^+$, where the Hessian is computed with respect the conformal

metric on M'.

5. The null generators of $\mathscr{I}^+$ are complete.
6. $\mathscr{I}^+$ is homeomorphic to $S^2 \times \mathbb{R}$, with the homeomorphism taking each null generator to a copy of $\mathbb{R}$.

These technical conditions, due to [GH78], ensure that $\mathscr{I}^+$ “looks like” the $\mathscr{I}^+$ for Minkowski space and is in this sense maximal. If the previous hypotheses also apply to $\mathscr{I}^-$, we say that M is asymptotically Minkowskian.

Asymptotically Euclidean initial data (M, g, K) can be defined in a similar way using conformal completions modeled on the conformal completion of Euclidean space into the sphere. We choose, however, to use an equivalent definition of asymptotically Euclidean data in terms of preferred charts near infinity and decay properties of g and K in these charts. Section Section 2.6 precisely defines asymptotically Euclidean initial data.

Given a spacelike hypersurface M of M, we count the number of black holes in M by counting the number of components of $\mathcal{B} \cap M$. Hence the number of black holes in a spacelike hypersurface (if any) is not a local property of the hypersurface, but a global property of the spacetime. This presents a significant challenge for constructing Cauchy data containing black holes. It would be nice to have a way to detect Cauchy data that will form a black hole without evolving the data, and this leads us to a discussion of trapped surfaces.

2.4 Trapped Surfaces and Apparent Horizons

Suppose Σ is a compact two-dimensional spacelike hypersurface in a four-dimensional Lorentzian manifold M. Then at each point $p \in \Sigma$ there exist a pair of future pointing normal vectors N_+ and N_- normal to Σ such that $T_p\mathsf{M} = T_p\Sigma \oplus \operatorname{span} N_+ \oplus \operatorname{span} N_-$. Since $\mathsf{g}(N_\pm, N_\pm) = 0$, we cannot use unit length normalization to select a distinguished choice of future pointing vector in $\operatorname{span} N_\pm$. The choice of vectors N_+ and N_- is unique, however, up to scaling and transposition. If the normal bundle of Σ is orientable, then N_+ and N_- can be extended to a pair of smooth future pointing null normal vector fields on Σ. Since M is time-orientable, the normal bundle of Σ is orientable if and only if there exists a globally defined spacelike unit normal vector ν to Σ. In particular, if Σ is the boundary of a spacelike hypersurface M, then the normal bundle of Σ is orientable. For simplicity we will assume that both the normal and tangent bundles of Σ are orientable.

We can construct a null hypersurface $\mathcal{N}_+$ containing Σ and ruled by null geodesics such that N_+ is tangent to these geodesics. The vector field N_+ on Σ extends naturally to a vector field, also called N_+, tangent to $\mathcal{N}_+$ and satisfying $\nabla_{N_+} N_+ = 0$ (here ∇ is the connection on the ambient Lorentzian manifold and N_+ is suitably extended off of $\mathcal{N}$;

such extensions will be made as needed without comment henceforth). On Σ we have the null second fundamental form χ_+ given by $\chi_+(X,Y) = -\langle \nabla_X N_+, Y\rangle$. Formally this expression resembles the second fundamental form of Riemannian geometry. However, since $\langle N_+, N_+\rangle = 0$, $\chi_+(X,Y)$ detects, in some sense, the part of $\nabla_X Y$ in $\operatorname{span} N_-$. The quantity $\theta_+ = -\operatorname{tr}\chi_+$ is known as the **convergence** or **expansion** of Σ with respect to N_+ and plays a similar role to mean curvature in Riemannian geometry. Using the flow of N_+ to define a family of spacelike surfaces Σ_t with $\Sigma_0 = \Sigma$ we see that θ_+ is well defined on $\mathcal{N}$, not just on Σ.

Now $\mathcal{N}$ possesses a unique globally defined two-form dA satisfying $dA(E_1, E_2) = 1$ for any pair of oriented orthonormal spacelike basis vectors to $\mathcal{N}$. An easy computation shows $\mathcal{L}_{N_+} dA = \theta_+ dA$. So θ_+ describes the change in area of Σ as it evolves under the flow of N_+. Since N_+ is defined only up to scale, it is interesting to note how θ_+ depends on N_+. The scale of N_+ defines a choice of affine parameter for the null geodesics that rule $\mathcal{N}$. If instead of N_+ we work with λN_+, where λ is a positive function on Σ, we obtain

$$\begin{aligned}
\chi_{N_+}(X,Y) &= \langle \nabla_X \lambda N_+, Y\rangle \\
&= X\lambda \langle N_+, Y\rangle + \lambda \langle \nabla_X N_+, Y\rangle \\
&= \lambda\, \chi_{N_+}(X,Y).
\end{aligned}$$

In particular, $\theta_{\lambda N_+} = \lambda \theta_{N_+}$ and hence the sign of θ_+ is geometrically significant. We say a surface Σ is **trapped** if both $\theta_\pm < 0$ and **marginally trapped** if both $\theta_\pm \leq 0$ on all of Σ. A trapped surface is (instantaneously) shrinking as it evolves under the flow of any family of orthogonal future pointing null geodesics.

To see how θ_+ evolves under the flow of N_+, we compute $\nabla_{N_+} \operatorname{tr}\chi_+ = \operatorname{tr}\nabla_{N_+}\chi_+$. Doing so, we arrive at

$$\begin{aligned}
\nabla_{N_+}\theta_+ &= -\left|\chi_+\right|^2 - \operatorname{Ric}(N_+, N_+) \\
&= -\frac{1}{2}\theta^2 - \left|\sigma_+\right|^2 - \operatorname{Ric}(N_+, N_+)
\end{aligned} \tag{5}$$

where σ_+ is the trace free part of χ_+ and Ric is the Ricci tensor of the ambient Lorentzian metric. Equation (5) is known as the Raychaudhuri equation. If $\operatorname{Ric}(N,N) \geq 0$ for all null vectors N (this is known as the **null energy condition**) then we can compare (5) with the ODE $x' = -x^2$. It follows from Gronwall's inequality that if $\theta_+ = \theta_0 < 0$ at affine parameter $t = 0$, then θ_+ tends to $-\infty$ somewhere within $t \in (0, -2/\theta_0]$. Points where $\theta_+ = -\infty$ are important because of a connection with conjugate points.

The family of null geodesics generating $\mathcal{N}_+$ is associated with a set of Jacobi fields arising from deviations through this family. A point p on an orthogonal null geodesic γ starting at Σ is said to be conjugate to Σ if there exists a nontrivial Jacobi field X arising from such a deviation that vanishes at p. It turns out that a necessary and sufficient condition for p to be conjugate to Σ is $\theta_+(p) = -\infty$. This implies that every null geodesic orthogonal to a trapped surface Σ has a conjugate point within a finite affine parameter from Σ (so long as the geodesic can be extended that far), and since Σ is compact we have a uniform bound on this parameter. One can also show that if p is conjugate to Σ, then the geodesic γ connecting p and Σ is homotopic to a *timelike* curve connecting p to Σ and hence $p \in I^+(\Sigma)$.

Trapped surfaces play an important role in the theory of gravitational collapse and the appearance of singularities. The following theorem of Penrose is typical. We sketch to proof to see the relationship between trapped surfaces, the appearance of conjugate points, and the appearance of singularities.

Theorem 2.2. [Pe65] *Let M be a spacetime satisfying the null energy condition. If M has a non-compact Cauchy surface M containing a trapped surface Σ, then there exists an inextendible null geodesic starting at Σ that terminates within finite affine parameter.*

Idea of Proof: Since M is globally hyperbolic, $J^+(\Sigma) = \overline{I^+(\Sigma)}$ and $\partial I^+(\Sigma)$ is ruled by null geodesics orthogonal to and terminating at Σ. Since Σ is trapped, each such geodesic has a conjugate point p within finite affine parameter hence $p \in I^+(\Sigma)$ and therefore $p \notin \partial I^+(\Sigma)$. It follows that $\partial I^+(\Sigma)$ is homeomorphic to a closed, bounded subset of $\Sigma \times \mathbb{R}$ and is hence compact. On the other hand, since M is a Cauchy surface we can establish a homeomorphism between $\partial I^+(\Sigma)$ and a subset S of M. Since $\partial I^+(\Sigma)$ is compact, so is S and in particular S is closed. Since $\partial I^+(\Sigma)$ is a manifold, S is locally Euclidean and hence open. Since M is connected, $S = M$, and since M is not compact we have a contradiction. □

In particular, Theorem 2.2 shows that if an asymptotically Euclidean initial data set contains a trapped surface, then the resulting maximal globally hyperbolic Cauchy development is not null complete. The theorem does not indicate the cause of the null incompleteness. One possibility that a singularity forms. Another is that the maximal development (M, g) can be embedded in a larger (not globally hyperbolic) spacetime $(\mathsf{M}', \mathsf{g}')$ In this second case, the boundary of M in M' is called a Cauchy horizon. There exists a stronger singularity theorem due to Hawking and Penrose [HP70] that does away with the Cauchy horizon possibility at the expense of adding more hypotheses. Regardless, a trapped surface is associated with pathological behaviour of the resulting Cauchy development. Now, a marginally trapped surface cannot be used in the previous proof since we have no guarantee that a marginally trapped surface generates null geodesics having conjugate points. On the other hand, we see from (5) that the condition $\theta_\pm = 0$ is not stable. For example, if $\chi_\pm \neq 0$ everywhere on Σ, then again we can conclude the existence of conjugate points. Hence marginally trapped surfaces are also of interest.

Another kind of surface, related to trapped surfaces, is an apparent horizon. Suppose Σ is contained in an asymptotically Euclidean Cauchy surface M and is the boundary of a region M_∞ of M containing spacelike infinity. We can then distinguish N_+ and N_- by the condition $\langle N_+, \nu \rangle < 0$, where ν is the outward pointing normal vector of M'. We say that such a Σ is **outer marginally trapped** if $\theta_+ \leq 0$ and is an **apparent horizon** if θ_+ vanishes identically. Apparent horizons are particularly interesting, since the boundary of a maximal foliation by trapped surfaces can be shown to be an apparent horizon, assuming the boundary is sufficiently smooth. In this sense, apparent horizons play a role for trapped surfaces that the event horizon does for the true boundary of a black hole.

One can show that trapped, marginally trapped and outer marginally trapped surfaces all signal the development of a black hole. This is important, since the expansion of a surface Σ in M can be computed directly using the initial data (M, g, K). Of course, the claim that a black hole appears is contingent on the resulting Cauchy evolution having an appropriate Scri. That this is true generically is known as the weak cosmic censorship which can be roughly formulated in the vacuum setting as follows.

Weak Cosmic Censorship Conjecture. *Let (g, K, M) be asymptotically Euclidean vacuum initial data satisfying appropriate smoothness and decay hypotheses. Then generically the maximal Cauchy evolution (M, g) of this data is asymptotically Minkowskian at future null infinity and $\mathsf{M} \cup \mathscr{I}^+$ is globally hyperbolic.*

The heuristic idea behind the conjecture is straightforward. If a complete $\mathscr{I}^+$ forms, observers sufficiently near infinity will never be affected by a pathology that develops in the spacetime. Moreover, since $\mathscr{I}^+$ is contained in the domain of dependence of M, no pathologies are visible at infinity. So if any exist, they must be contained in a black hole. The generic caveat is present in the conjecture since it is know that certain spherically symmetric initial data for gravity coupled with a Klein-Gordon matter field do form naked singularities, but that these singularities do not persist under perturbations. In the vacuum setting, no known counterexamples to weak cosmic censorship exist. We have the following theorem relating trapped surfaces and black holes

Theorem 2.4. [HE73][Wa84] *Let (g, K, M) be initial data containing a trapped, marginally trapped, or marginally outer trapped surface Σ. If the maximal Cauchy development (M, g) of the data is weakly censored, then $I^-(\mathscr{I}^+) \cap \Sigma = \emptyset$ and hence Σ is contained in the black hole region of (M, g).*

Although this result is widely accepted in the relativity community, the proofs in the cited texts are not correct (except for [Wa84] in the context of trapped surfaces and with mildly stronger hypotheses than those stated here). One can, in fact, give a correct proof for trapped surfaces. The more delicate case of marginally (outer) trapped surfaces still needs to be addressed, but it seems likely a correct proof of this result can be found here also[Ch03].

2.5 Initial Data Containing Black Holes

Theorem 2.4 motivates finding finding solutions (M, g, K) of the constraint equations (3) and (4) containing trapped surfaces or apparent horizons. Let (M, g, K) be an asymptotically Euclidean data set, and let Σ be an orientable compact hypersurface with unit normal ν. If M is a Cauchy surface for a spacetime M with future pointing timelike unit normal n, then the vectors $N_{\pm} = \pm\nu + n$ are null, future pointing, and orthogonal to ∂M. The convergences $\theta_{\pm}$ computed with respect to $N_{\pm}$ are

$$\theta_{\pm} = -\operatorname{tr} K + K(\nu, \nu) \mp 2h,$$

where h is the mean curvature of ∂M in M computed with respect to ν. For time symmetric initial data (that is data with $K = 0$), then the equation $\theta_{+} = 0$ reduces to $h = 0$. The Hamiltonian constraint (3) reduces to $R = 0$ and the momentum constraint (4) is satisfied automatically. So a time symmetric solutions of the constraints with an apparent horizon is just a scalar flat Riemannian manifold with a minimal surface. Consider the manifold $\mathbb{R} \times S^2$ with the metric $(1 + s^2)ds^2 + \left(\frac{s^2+1}{2}\right)^2 dS^2$, where s is the coordinate along $\mathbb{R}$ and dS^2 is the round metric on the sphere. One readily shows this manifold is scalar flat and that the level set $s = 0$ is a minimal surface. The maximal development of this data is a member the one parameter family of Schwarzschild solutions of the Einstein equation, and is a prototypical black hole solution.

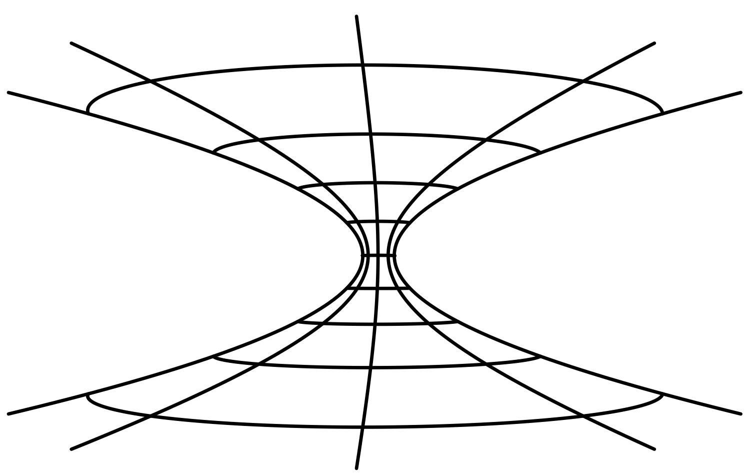

Figure 2: Time Symmetric Slice of Schwarzschild

Note from Figure 2 that the Schwarzschild initial data has two asymptotically Euclidean ends connected by a neck. For the purposes of computing the black hole region $\mathcal{B}$ of the resulting Cauchy development, we select a distinguished end and work with the Scri of that end.

One can imagine similar data formed by connecting two asymptotically Euclidean regions together with several necks, or connecting several asymptotically Euclidean regions to a given distinguished one. Schemes such as those in in [Mi63], [BL63], [YB80] (see also [Ck00]) create families of initial data containing apparent horizons or trapped surfaces inside necks. A very flexible approach for generating necks comes from a gluing construction [IMP02]. One can, for example, start with two asymptotically Euclidean solutions of the constraints (M_i, g_i, K_i), $i = 1, 2$, and generate a third solution $(M_1 \# M_2, g, K)$ on the connected sum. The new solution will contain a neck and will closely approximate the original solutions away from the surgery location. In certain cases, one can prove rigorously that necks introduced this way will evolve into distinct black holes [CM03].

The previous methods for generating black hole initial data create apparent horizons indirectly by topological means. A direct approach for creating apparent horizons, introduced by Thornberg [Th87], is to work with a manifold with boundary and prescribing that the boundary be an apparent horizon. Thornburg numerically investigated generating such initial data, and variations of the apparent horizon condition have subsequently been proposed for numerical study, e.g. [Ck02] [Ea98]. However, as indicated by Dain [Da02], there has not been a rigorous mathematical investigation of the apparent horizon boundary condition.

Let (M, g, K) be an asymptotically Euclidean data set on a manifold with compact boundary, and let ν denote the exterior unit normal to ∂M. The convergences $\theta_\pm$ computed at the boundary are

$$\theta_\pm = -\operatorname{tr} K + K(\nu, \nu) \mp 2h,$$

where h is the mean curvature of ∂M in M computed with respect to $-\nu$. The convergence θ_+ corresponds to the outgoing (to infinity) null direction and hence the boundary is an apparent horizon if $\theta_+ = 0$. So our goal is to find initial data satisfying

$$\begin{aligned} R - |K|^2 + \operatorname{tr} K^2 &= 0 \\ \operatorname{div} K - d \operatorname{tr} K &= 0 \\ -\operatorname{tr} K + K(\nu, \nu) - 2h &= 0 \quad \text{on } \partial M. \end{aligned} \tag{6}$$

The content of Section 3 is a construction of solutions of this boundary value problem. Before proceeding with the construction, we first make precise the function spaces used.

2.6 Asymptotically Euclidean Manifolds and Weighted Function Spaces

It was mentioned in Section 2 that asymptotically Euclidean manifolds an be defined in terms of conformal completions. We find it convenient to take a coordinate based approach more amenable to analysis. An asymptotically Euclidean manifold is a non-compact Riemannian manifold, possibly with boundary, that can be decomposed into a compact core and a finite number of ends $\{N_i\}_{i=1}^m$. Each end N_i is diffeomorphic to the region exterior to the closed unit ball in $\mathbb{R}^n$, and the metric on N_i is asymptotic to the Euclidean metric at far distances.

To make this loose description precise, we use weighted function spaces that prescribe asymptotic behavior like $|x|^\delta$ for large x. For $x \in \mathbb{R}^n$, let $w(x) = (1+|x|^2)^{1/2}$. Then for any $\delta \in \mathbb{R}$ and and any open set $\Omega \subset \mathbb{R}^n$, the weighted Sobolev norm on functions or tensor fields over Ω is

$$||u||_{W^{k,p}_\delta(\Omega)} = \sum_{|\beta|\le k} ||w^{-\delta-\frac{n}{p}+|\beta|}\,\partial^\beta u||_{L^p(\Omega)}.$$

Unless explicitly stated otherwise, we always assume Lebesgue exponents are neither 1 nor infinity. Typically Ω is either all of $\mathbb{R}^n$ or is an exterior region $E_r = \{x \in \mathbb{R}^n : |x| > r\}$. The weighted Sobolev space $W^{k,p}_\delta(\Omega)$ is the subset of $W^{k,p}_{\mathrm{loc}}(\Omega)$ for which the weighted Sobolev norm is finite. Our indexing convention for δ follows [Ba86] so that the value of δ directly encodes asymptotic growth at infinity.

The weighted space $C^k_\delta(\Omega)$ is defined similarly as the subset of $C^k(\Omega)$ such that the weighted norm

$$||u||_{C^k_\delta(\Omega)} = \sum_{|\alpha|\le k} \sup_{x\in\Omega} w(x)^{-\delta+|\alpha|}\,|\partial^\alpha u(x)|$$

is finite.

We have the following facts concerning weighted Sobolev spaces.

Lemma 2.5.

1. *If $p \le q$ and $\delta' < \delta$ then $L^p_{\delta'}(\Omega) \subset L^q_\delta(\Omega)$ and the inclusion is continuous.*
2. *For $k \ge 1$ and $\delta' < \delta$ the inclusion $W^{k,p}_{\delta'}(\Omega) \subset W^{k-1,p}_\delta(\Omega)$ is compact.*
3. *If $1/p > k/n$ then $W^{k,p}_\delta(\Omega) \subset L^r_\delta(\Omega)$ where $1/r = 1/p - k/n$. If $1/p = k/n$ then $W^{k,p}_\delta(\Omega) \subset L^r_\delta(\Omega)$ for all $r \ge p$. If $1/p < k/n$ then $W^{k,p}_\delta(\Omega) \subset C^0_\delta(\Omega)$. These inclusions are continuous.*
4. *If $m \le \min(j,k)$, $p \le q$, $\epsilon > 0$, and $1/q < (j+k-m)/n$, then multiplication is a continuous bilinear map from $W^{j,q}_{\delta_1}(\Omega) \times W^{k,p}_{\delta_2}(\Omega)$ to $W^{m,p}_{\delta_1+\delta_2-\epsilon}(\Omega)$. In particular, if $1/p < k/n$ and $\delta < 0$, then $W^{k,p}_\delta(\Omega)$ is an algebra.*
5. *If $p \in [1,n)$ and $\delta = 1 - n/p$, then $W^{1,p}_\delta(E_1)$ is continuously embedded in*

$L^{p^*}(E_1)$.

Let M be a smooth, connected, n-dimensional manifold with boundary, and let g be a metric on M for which (M, g) is complete (these will be standing assumptions for the remainder of the paper). We say (M, g) is **asymptotically Euclidean of class** $W^{k,p}_\rho$ if:

i. The metric $g \in W^{k,p}_{\text{loc}}(M)$, where $1/p - k/n < 0$ (and consequently g is continuous).

ii. There exists a finite collection $\{N_i\}_{i=1}^m$ of open subsets of M and diffeomorphisms $\Phi_i : E_1 \mapsto N_i$ such that $M - \cup_i N_i$ is compact.

iii. For each i, $\Phi_i^* g - \overline{g} \in W^{k,p}_\rho(E_1)$, where $\rho < 0$ and $\overline{g}$ is the Euclidean metric.

We define asymptotically Euclidean manifolds of class C^k_ρ analogously. Henceforth ρ will denote a negative number. Note that the boundary of an asymptotically Euclidean manifold is necessarily compact.

The charts Φ_i are called end charts and the corresponding coordinates are end coordinates. Suppose (M, g) is asymptotically Euclidean, and let $\{\Phi_i\}_{i=1}^m$ be its collection of end charts. Let $K = M - \cup_i \Phi_i(E_2)$, so K is a compact manifold with boundary. The weighted Sobolev space $W^{k,p}_\delta(M)$ is the subset of $W^{k,p}_{\text{loc}}(M)$ such that the norm

$$||u||_{W^{k,p}_\delta(M)} = ||u||_{W^{k,p}(K)} + \sum_i ||\Phi_i^* u||_{W^{k,p}_\delta(E_1)}$$

is finite. The weighted spaces $L^p_\delta(M)$ and $C^k_\delta(M)$ are defined similarly, and we let $C^\infty_\delta(M) = \cap_{k=0}^\infty C^k_\delta(M)$. Lemma Lemma 2.5 applies equally well to asymptotically Euclidean manifolds, and Lemma 2.5 implies $W^{1,p}_\delta(M) \subset L^{p^*}(M)$ for $p \in [1, n)$ and $\delta = 1 - n/p$.

Using these weighted spaces we can now define an asymptotically Euclidean data set. The extrinsic curvature tensor K of an initial data set (M, g, K) should behave like a first derivative of g. Hence, if (M, g) is asymptotically Euclidean of class $W^{k,p}_\rho$, we say (M, g, K) is an asymptotically Euclidean data set if $K \in W^{k-1,p}_{\rho-1}(M)$.

3. Construction of Solutions of the Boundary Value Problem

3.1 CMC Solutions of the Constraint Equations on Asymptotically Euclidean Manifolds Without Boundary

Before treating the boundary value problem, we first recall a method for constructing solutions of the constraint equations on manifolds without boundary. A constant mean curvature (CMC) solution (M, g, K) of the constraint equations is one for which $\operatorname{tr} K$ is constant ($\operatorname{tr} K$ is the mean curvature of M in its ambient Cauchy development). The CMC conformal method, due to Lichnerowicz [Li44] and Choquet-Bruhat and York [CBY80], provides an effective tool for constructing these solutions. We start with a Riemannian manifold (M, g), a traceless, symmetric $(0,2)$-tensor σ, and a constant τ. We then seek a solution of the constraints $(M, \hat{g}, \hat{K})$ where

$$\begin{aligned} \hat{g} &= \phi^{\frac{4}{n-2}} g \\ \hat{K} &= \phi^{-2}\sigma + \frac{\tau}{n}\hat{g} \end{aligned} \tag{7}$$

(here n is the dimension of M, which is henceforth assumed to be at least 3). One readily verifies that $\widehat{\operatorname{div}}\phi^{-2}\sigma = \phi^{-\frac{2n}{n-2}} \operatorname{div}\sigma$ and $\widehat{\operatorname{tr}}\hat{K} = \tau$. In the asymptotically Euclidean setting, we have a further simplification. Decay conditions on $\hat{K}$ at infinity imply that if $\widehat{\operatorname{tr}}\hat{K}$ is constant, then it must be zero. So we assume from the start that $\tau = 0$.

Using conformal covariance, the left-hand side of the momentum constraint (4) becomes

$$\widehat{\operatorname{div}}\hat{K} - d\,\widehat{\operatorname{tr}}\hat{K} = \widehat{\operatorname{div}}\phi^{-2}\sigma = \phi^{-\frac{2n}{n-2}} \operatorname{div}\sigma.$$

So (4) reduces simply to

$$\operatorname{div}\sigma = 0. \tag{8}$$

A trace-free, symmetric $(0,2)$-tensor σ satisfying (8) is called **transverse traceless**.

The Hamiltonian constraint on $(M, \hat{g}, \hat{K})$ written in terms of (M, g, σ, ϕ) then becomes

$$-\Delta\,\phi + \frac{1}{a}\left(R\phi - |\sigma|^2\,\phi^{-3-2\kappa}\right) = 0, \tag{9}$$

where $\kappa = 2/(n-2)$ and $a = 2\kappa + 4$. Equation (9) is known as the Lichnerowicz equation. Since ϕ is a conformal factor, we require $\phi > 0$. To ensure that $(M, \hat{g})$ is asymptotically Euclidean, we also impose a condition at infinity, requiring $\phi - 1 \in C^0_\delta$ for some $\delta < 0$.

Now the set of transverse traceless tensors forms a linear space, and the choice of σ can be thought of as data to be prescribed in solving (9). It remains, then, to find conditions on (M, g, σ) under which (9) is solvable.

From (3) if follows that if (9) has a solution, then g has a conformally related asymptotically Euclidean metric $\hat{g}$ such that $\hat{R} = |\hat{K}|^2 \geq 0$. It follows that

$$\int_M a\left|\widehat{\nabla} f\right|^2 + \hat{R} f^2\,\widehat{dV} > 0$$

for all smooth, compactly supported functions f not identically 0. Now for fixed f, the sign of the previous integral is a conformal invariant and it follows that

$$\int_M a\,|\nabla f|^2 + R f^2\,dV > 0 \quad \text{for all } f \in C^\infty_c(M),\ f \not\equiv 0 \tag{10}$$

as well. So (10) is a necessary condition for (9) to be solvable. It was previously reported [CaB81] [CBIY00] that (10) is also sufficient. One of the results of [Ma03] is that this isn't quite true, and the correct condition is

$$\lambda_g = \inf_{f \in C^\infty_c(M), f \not\equiv 0} \frac{\int_M a \left|\nabla f\right|^2 + R f^2 \, dV}{||f||^2_{L^{2n/(n-2)}}} > 0. \tag{11}$$

The quantity λ_g is a conformal invariant related to the Yamabe invariant in the compact setting. One must be careful with this analogy however. For example, if $\lambda_g > 0$, then g is conformally related to a metric with positive scalar curvature, to a metric with negative scalar curvature, and also to a scalar flat metric.

To solve (9) we note that if $\lambda_g > 0$, then we can assume without loss of generality that $R = 0$. Letting $\phi = 1 + v$ we find (9) becomes

$$-\Delta\, v = \left|\sigma\right|^2 (1+v)^{-3-2\kappa}. \tag{12}$$

Now $v_- = 0$ is a subsolution of (12) (i.e $-\Delta\, v_- \leq \left|\sigma\right|^2 (1+v_-)^{-3-2\kappa}$). Moreover, letting v_+ be the solution of

$$-\Delta\, v_+ = \left|\sigma\right|^2 \tag{13}$$

that decays at infinity, it follows from a maximum principle argument that $v_+ \geq 0$. Since $\left|\sigma\right|^2 (1+v_+)^{-3-2\kappa} \leq \left|\sigma\right|^2$ we have v_+ is a supersolution of (13). Since $v_- \leq v_+$, a barrier argument (first introduced in the context of the constraint equations on compact manifolds in [Is95]) shows that there exists a solution of (13). One can also show that this solution is unique. Stated precisely in terms of weighted function spaces, we have the following theorem, taking into account the correction in [Ma03].

Theorem 3.1. [Ca77][CaB81][CBIY00] *Let (M, g) be an asymptotically Euclidean manifold of class $W^{k,p}_\delta$ with $k > n/p + 2$ and $\delta \in (n-2, 0)$, and let $\sigma \in W^{k-1,p}_{\delta-1}(M)$ be a transverse traceless tensor. Then there exists a solution $\phi > 0$ of (9) with $1 - \phi \in W^{k,p}_\delta$ if and only if $\lambda_g > 0$, and any such solution is unique.*

We have an equivalence class on data (M, g, σ) from the relationship $(M, g_1, \sigma_1) \sim (M, g_2, \sigma_2)$ if there exists a conformal factor ϕ with $g_1 = \phi^{2\kappa} g_2$ and $\sigma_1 = \phi^{-2} g_2$. Hence the set of CMC asymptotically Euclidean solutions of the constraint equations on a manifold M is parameterized by equivalence classes $[M, g, \sigma]$ where $\lambda_g > 0$.

The restriction $k > n/p + 2$ in Theorem 3.1 arises in [CBIY00] to ensure that g has two Hölder continuous derivatives. In light of recent low regularity a priori estimates for solutions of the evolution problem [KR02] [ST], there is interest in generating low regularity solutions of the constraint equations. These papers provide estimates (when $n = 3$) for initial data (g, K) in $H^{2+\epsilon}_{\mathrm{loc}} \times H^{1+\epsilon}_{\mathrm{loc}}$.

A secondary result of [Ma03] is the construction of asymptotically Euclidean solutions of the constraints with $k \geq 2$ and $k > n/p$,[†] so long as (M, g) has no conformal Killing fields vanishing at infinity. The lower bound $k \geq 2, k > n/p$ is the weakest regularity that ensures that g has curvature in an L^p space and that the Sobolev space containing g is an algebra. When $n = 3$, this implies the existence of $H^2_{\rm loc} \times H^2_{\rm loc}$ solutions (and from standard elliptic regularity arguments $H^{2+\epsilon}_{\rm loc} \times H^{1+\epsilon}_{\rm loc}$ solutions as well).

3.2 Solving the Boundary Value Problem

Our goal is to adapt the CMC conformal method to manifolds with boundary. As in the case of no boundary, we start with an asymptotically Euclidean manifold (M, g) and a transverse traceless tensor σ. We then seek to find a conformal factor ϕ such that $\hat{g} = \phi^\kappa g$ and $\hat{K} = \phi^{-2}\sigma$ solves the constraint equations, and we want ∂M to satisfy $\hat{\theta}_+ = 0$. The convergences $\hat{\theta}_\pm$ can be written in terms of 'unhatted' variables, and in particular the condition $\hat{\theta}_+ = 0$ becomes

$$\phi^{-1-\kappa}\partial_\nu \phi + \frac{1}{\kappa} h \phi^{-\kappa} - \frac{2}{a}\phi^{-2-2\kappa}\sigma(\nu,\nu) = 0.$$

So we want to find conditions on (M, g, σ) under which the boundary value problem

$$\begin{aligned} -\Delta\,\phi + \frac{1}{a}\left(R\phi - |\sigma|^2\,\phi^{-3-2\kappa}\right) &= 0 \\ \partial_\nu\phi + \frac{1}{\kappa}h\phi - \frac{2}{a}\phi^{-1-\kappa}\sigma(\nu,\nu) &= 0 \quad \text{on } \partial M \end{aligned} \tag{14}$$

is solvable.

Given the analysis of the case $\partial M = \emptyset$, it seems reasonable that there will be a restriction on the conformal class $[g]$. For compact manifolds with boundary, there is a conformal invariant related to the Yamabe invariant [Es92]. Translating this invariant into the asymptotically Euclidean setting we define

$$\lambda_g = \inf_{f \in C^\infty_c(M), f \not\equiv 0} \frac{\int_M a\,|\nabla f|^2 + Rf^2\; dV + \int_{\partial M} \frac{a}{\kappa} h f^2\; dA}{||f||^2_{L^{2n/(n-2)}}}.$$

The main result (Theorem 4.3) of [Ma03] shows that (14) is solvable if

1. (M, g) satisfies $\lambda_g > 0$, $R = 0$, and $h < 0$.
2. σ satisfies $(n-1)h \leq \sigma(\nu,\nu) \leq 0$.

[†] Y. Choquet-Bruhat has announced a construction of similar low regularity solutions of the constraint equations in the context of compact manifolds.

For manifolds without boundary, it is obvious that $\lambda_g > 0$ is a necessary condition. Although it is not clear if this condition is also necessary to solve the boundary value problem (14), the construction from [Ma03] requires it because it ensures

$$\mathcal{P} = (-\Delta\ + \frac{1}{a}R) \otimes (\partial_\nu + \frac{1}{\kappa}h)$$

is an isomorphism acting on certain weighted Sobolev spaces. Now given any asymptotically Euclidean manifold (M, g') with $\lambda_{g'} > 0$, we can in fact conformally change to an asymptotically Euclidean manifold satisfying $R = 0$ and $h < 0$ [Ma03]. So in some sense the requirements $R = 0$ and $h < 0$ are superfluous. We make these requirements explicit, however, since condition 2 is not conformally invariant; the inequality in condition 2 must hold with respect to a conformal representative having $R = 0$ and $h < 0$.

To motivate condition 2, we first consider the sign condition $\sigma(\nu,\nu) \leq 0$. Since $\hat{K}(\hat{\nu},\hat{\nu}) = \phi^{-2\kappa-2}\sigma(\nu,\nu)$, it follows that the sign of $\sigma(\nu,\nu)$ determines the sign of $\hat{K}(\hat{\nu},\hat{\nu})$. Now if $\hat{\theta}_+ = 0$, it follows that

$$\hat{K}(\hat{\nu},\hat{\nu}) = (n-1)\hat{h}.$$

Thus the sign of $\sigma(\nu,\nu)$ also determines the sign of $\hat{h}$. Finally, the sign of $\hat{h}$ determines a relationship between $\hat{\theta}_+$ and $\hat{\theta}_-$. Since

$$\begin{aligned}\hat{\theta}_+ &= \hat{K}(\hat{\nu},\hat{\nu}) - (n-1)\hat{h}\\ \hat{\theta}_- &= \hat{K}(\hat{\nu},\hat{\nu}) + (n-1)\hat{h},\end{aligned}$$

we conclude that $\hat{h} \leq 0$ implies $\hat{\theta}_+ \geq \hat{\theta}_-$ whereas $\hat{h} \geq 0$ implies $\hat{\theta}_+ \leq \hat{\theta}_-$. Hence we require $\sigma(\nu,\nu) \leq 0$ to ensure $\hat{\theta}_- \leq \hat{\theta}_+ = 0$ and therefore the boundary of M is not only an apparent horizon, but also a marginally trapped surface. In fact, an earlier version of [Ma03] worked with the condition $\sigma(\nu,\nu) \geq 0$. Although this allowed for the construction of apparent horizons, it was observed in [Da03] that these surfaces are of limited physical interest because they are not marginally trapped surfaces. For comparison, we consider the approach of [Da03]. Rather than work with $\hat{\theta}_+$, Dain prescribes $\hat{\theta}_- \leq 0$ and under suitable conditions constructs solutions satisfying $\hat{\theta}_+ \leq \hat{\theta}_- \leq 0$. These are trapped surfaces, but the relationship $\hat{\theta}_+ \leq \hat{\theta}_-$ shows that for these solutions $\hat{h} \geq 0$. Hence the method of [Da03] cannot construct an apparent horizon (except in the extremal case $\hat{\theta}_+ = \hat{\theta}_- = \hat{h} = 0$). To create an apparent horizon that is also a marginally trapped surface, we must have $\hat{h} \leq 0$. Figure 3 shows boundaries with mean curvatures of different signs and indicates the difference between the conditions $\hat{h} \geq 0$ and $\hat{h} \leq 0$.

We now analyze to the condition $(n-1)h \leq \sigma(\nu,\nu)$. Since $\sigma(\nu,\nu) \leq 0$, we have the necessary consequence $h \leq 0$. From an analysis point of view, we would rather have $h \geq 0$. For example, if $R \geq 0$ and $h \geq 0$, then there is a maximum principle associated with $\mathcal{P}$. This would be a useful tool to show conformal factors we construct are positive.

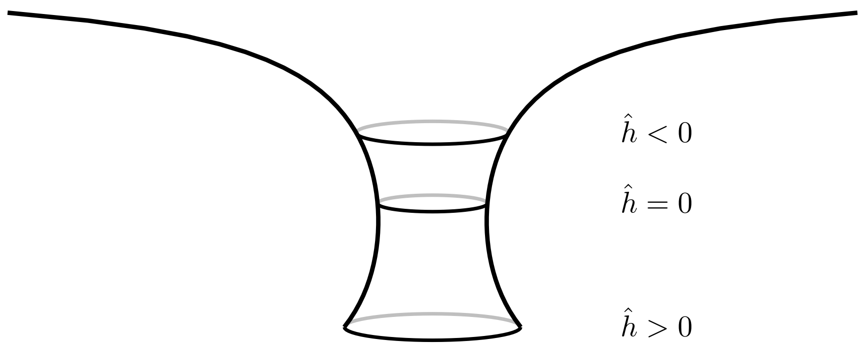

Figure 3: Boundary Mean Curvatures of an Asymptotically Euclidean Manifold

This last fact is part of the motivation for the condition $\sigma(\nu,\nu) \geq 0$ in the prior version of [Ma03] and also for the choice in [Da03] to work with $\hat{\theta}_-$ rather than $\hat{\theta}_+$. The inequality $(n-1)h \leq \sigma(\nu,\nu)$ is used to compensate for the loss of the maximum principle. Since it is not conformally invariant, it is certainly the least appealing condition of the construction. We note, however, that the equality $(n-1)h = \sigma(\nu,\nu)$ is exactly the condition $\theta_+ = 0$, giving some insight into the meaning of this condition.

3.3 Manifolds of the correct conformal class

For a compact Riemannian manifold (M,g) without boundary, there is a well known relationship between the Yamabe invariant, a geometric condition on (M,g), and an analytic condition on (M,g). Namely, the following are equivalent.

1. There is a metric $\tilde{g} \in [g]$ with scalar curvature everywhere positive (resp. negative, zero).
2. The Yamabe invariant λ_g is positive (resp. negative, zero).
3. The first non-zero eigenvalue of $-\Delta_g + \frac{1}{a}R$ is positive (resp. negative, zero).

In the asymptotically Euclidean setting, we can also show that the condition $\lambda_g > 0$ is equivalent to a geometric condition and an analytic condition. Since the conformal Laplacian does not have eigenfunctions that vanish at infinity, the analytic condition cannot be expressed in terms of an eigenvalue. To state it, we consider instead the family of operators

$$\mathcal{P}_\eta = (-\Delta + \frac{\eta}{a}R) \otimes (\partial_\nu + \frac{\eta}{\kappa}h)\Big|_{\partial M}.$$

[Ma03] Proposition 4.1. *Suppose (M,g) is asymptotically Euclidean of class $W^{k,p}_\delta$, $k \geq 2$, $k > n/p$, and $2-n < \delta < 0$. Then the following conditions are equivalent:*

1. *There exists a conformal factor* $\phi > 0$ *such that* $1 - \phi \in W^{k,p}_\delta$ *and such that* $(M, \phi^{2\kappa} g)$ *is scalar flat and has a minimal surface boundary.*

2. $\lambda_g > 0$.

3. $\mathcal{P}_\eta$ *is an isomorphism acting on* $W^{k,p}_\delta$ *for each* $\eta \in [0,1]$.

The role of η becomes clear in the implication $3 \Rightarrow 1$. Assuming 3 is true, we seek a conformal factor $\phi = 1 + v$ with $v \in W^{k,p}_\delta$ solving $\mathcal{P}_1 v = (-\frac{R}{a}, -\frac{h}{\kappa})$. Although we can solve for v, we also want to show that $\phi = 1 + v > 0$. To do this we consider a family of functions $\phi_\eta = 1 + v_\eta$ where v_η solves $\mathcal{P}_\eta v_\eta = (-\frac{\eta R}{a}, -\frac{\eta h}{\kappa})$. Starting with $\eta = 0$ we have a solution $v_0 = 0$ and hence $\phi_0 > 0$. Moving through the family v_η we prove that ϕ_η can never pass through zero by means of a Harnack inequality, and in particular $1 + v_1 > 0$. The spirit of this proposition comes from a related proof in [CaB81] for manifold without boundary (A version of [Ma03] Proposition 4.1 holds for manifolds without boundary by dropping the boundary terms in the definitions of λ_g and $\mathcal{P}_\eta$). In [CaB81] there is an erroneous proof that condition 1 is equivalent to

$$\int_M a\,|\nabla f|^2 + Rf^2\,dV > 0 \quad \text{for all } f \in C^\infty_c(M)\,,\, f \not\equiv 0. \tag{15}$$

This condition is necessary, but not sufficient, and the normalization in the definition λ_g allows for sufficiency.

It is reasonable to ask if any manifolds with boundary satisfy $\lambda_g > 0$. Given [Ma03] Proposition 4.1 it is enough to find manifolds (M,g) with $R \geq 0$ and $h \geq 0$ (a maximum principle then implies condition 3). First, suppose we have an asymptotically Euclidean manifold without boundary (M', g') with $\lambda_{g'} > 0$. Examples of this include $\mathbb{R}^n$ with the flat metric (or nearby metrics since $\lambda_{g'} > 0$ is an open condition) or any maximal solution of the constraint equations. From [Ma03] Proposition 4.1 we can assume that $R' = 0$. Let G be the Greens function for the conformal Laplacian on M' with singularity at x. From condition 3 it exists, from the maximum principle it is positive, and it has a singularity of order r^{2-n} at x.

Let $\phi = 1 + G$ and let $g = \phi^{2\kappa} g'$ on $M = M' - B_\epsilon(X)$. We wish to show that if ϵ is small enough, then $h > 0$. Now $G = r^{2-n} + O(r^{3-n})$ and $h' = -r^{-1} + O(1)$. From the conformal change of mean curvature we have

$$\begin{aligned} h &= \phi^{-\kappa-1}\left(-\partial_r r^{2-n} - \frac{1}{\kappa} r^{2-n}\frac{1}{r} + O(r^{2-n})\right)\Big|_{r=\epsilon} \\ &= \phi(\epsilon)^{-\kappa-1}\left(\frac{1}{\kappa}\epsilon^{1-n} + O(\epsilon^{2-n})\right) \\ &> 0 \end{aligned}$$

Repeating this argument (augmenting it in the obvious way to accommodate the boundary) we can remove another small ball from this manifold, and so on to create as many boundary components as we please. Hence there is a rich collection of manifolds with $\lambda_g > 0$.

3.4 The conformal transformation

The next step of the construction is to show that given a manifold (M, g') with $\lambda_{g'} > 0$, we can conformally transform to (M, g) with $R = 0$ and $h < 0$.

[Ma03] Corollary 4.2. *Suppose (M, g') is asymptotically Euclidean of class $W^{k,p}_\delta$, $k \geq 2$, $k > n/p$, and $2 - n < \delta < 0$. If $\lambda_{g'} > 0$, then there exists a conformal factor $\phi > 0$ such that $1 - \phi \in W^{k,p}_\delta$ and such that $(M, g) = (M, \phi^{2\kappa} g')$ is scalar flat, has negative boundary mean curvature, and satisfies $\lambda_g > 0$.*

In fact, we could even have stipulated $R < 0$ as well. This might be surprising given that the condition $\lambda_g > 0$ on a compact manifold without boundary would preclude a change to a manifold with everywhere negative scalar curvature. The asymptotically Euclidean end yields greater flexibility, and to prove [Ma03] Corollary 4.2 we merely solve

$$\begin{aligned} -\Delta_{g'} v_\epsilon &= 0 \\ \partial_{\nu'} v_\epsilon &= -\epsilon. \end{aligned}$$

Taking ϵ close to 0 yields a v_ϵ close to zero, and $\phi_\epsilon = 1 + v_\epsilon$ is then a genuine conformal factor. From the conformal transformation of scalar and mean curvatures it follows that $g = \phi_\epsilon^{2\kappa} g'$ has $R = 0$ and $h < 0$.

3.5 Appropriate transverse traceless tensors

At this stage of the construction we have a manifold (M, g) satisfying $\lambda_g > 0$, $R = 0$, and $h < 0$. The next step is to find a transverse traceless tensor σ satisfying $(n-1)h \leq \sigma(\nu, \nu) \leq 0$. To do this, we consider the conformal killing operator $\mathbb{L}$, where $\mathbb{L}X = \frac{1}{2}\mathcal{L}X g - \frac{1}{n}(\operatorname{div} X)g$. Then the vector Laplacian $\Delta_{\mathbb{L}} = \operatorname{div} \mathbb{L}$ is an elliptic operator on M, and the Neumann boundary operator B corresponding to $\Delta_{\mathbb{L}}$ takes a vector field X to the covector field $\mathbb{L}X(\nu, \cdot)$. We propose to solve the boundary value problem

$$\begin{aligned} \Delta_{\mathbb{L}} X &= 0 \\ BX &= \omega \quad \text{on } \partial M, \end{aligned} \tag{16}$$

where ω is a covector field over ∂M. If we can do this, then letting $\sigma = \mathbb{L}X$ it follows that σ is trace and divergence free. Moreover, $\sigma(\nu, \nu) = \omega(\nu)$ on ∂M, so taking ω such that $(n-1)h \leq \omega(\nu) \leq 0$ ensures $(n-1)h \leq \sigma(\nu, \nu) \leq 0$.

Let $\mathcal{P}^{k,p}_\delta$ denote $\mathbb{L} \otimes B$ acting as a map from $W^{k,p}_\delta(M)$ to $W^{k-2,p}_{\delta-2}(M) \times W^{k-1-\frac{1}{p},p}(\partial M)$. We have the following mapping property of $\mathcal{P}^{k,p}_\delta$.

[Ma03] Proposition 5.4. *Suppose* (M,g) *is asymptotically Euclidean of class* $W^{k,p}_\rho$ *with* $k > n/p$, $k \geq 2$*, and suppose* $2-n < \delta < 0$*. Then* $\mathcal{P}^{k,p}_\delta$ *is Fredholm of index 0. Moreover, it is an isomorphism if and only if* (M,g) *possesses no nontrivial conformal Killing fields* X *in* $W^{k,p}_\delta(M)$ *satisfying* $BX = 0$.

The proof of this theorem comes from a careful reworking of the results of [CBC81] taking into account the boundary and the choice to work with low regularity spaces (the theorems in in [CBC81] require $k > \frac{n}{p}+1$). The lower bound $k > \frac{n}{p}$ in [Ma03] Proposition 5.4 arises to ensure Hölder continuity of the leading order coefficients and thereby permit a coefficient freezing argument to go through. The key step in the proof of [Ma03] Proposition 5.4 is proving the a priori coercivity estimate

$$||X||_{W^{k,p}_\delta(M)} \lesssim ||\Delta_{\mathbb{L}} X||_{W^{k-2,p}_{\delta-2}(M)} + ||B\,X||_{W^{k-1-\frac{1}{p},p}(\partial M)} + ||X||_{L^p_{\delta'}(M)}. \tag{17}$$

Since [Ma03] Proposition 5.4 reduces the question of $\mathcal{P}^{k,p}_\delta$ being an isomorphism to the existence of certain conformal killing fields on (M,g), it would be nice to rule out their existence. Now it is known [CO81] that if (M,g) is asymptotically Euclidean of class C^3, then there are no conformal Killing fields decaying at infinity (the arguments of [CO81] are given for manifolds without boundary, but extend trivially to manifolds with boundary as well). So we are lead to hypothesize there are no conformal Killing fields vanishing at infinity even for low regularity metrics. The following result is true.

[Ma03] Theorem 6.4. *Suppose* (M,g) *is asymptotically Euclidean of class* $W^{2,p}_\rho$ *with* $p > n$*. Then there exist no nontrivial conformal Killing fields in* $W^{2,p}_\delta$ *for any* $\delta < 0$.

The proof of [Ma03] Theorem 6.4 is by blowup argument and by analysis of conformal killing fields on $\mathbb{R}^n$. We first do a blowup argument at infinity to construct a conformal killing field on $\mathbb{R}^n$ vanishing at infinity. It is easy to see any such conformal Killing field is trivial. We then do a blowup argument on the boundary of this open set and prove from analyzing the limit that in fact the open set must be the interior of M.

It follows from [Ma03] Theorem 6.4 and Sobolev embedding that if (M,g) is asymptotically Euclidean of class $W^{k,p}_\rho$ with $k > \frac{n}{p}+1$ that there are no conformal Killing fields vanishing at infinity. We would like to see this reduced to $k > \frac{n}{p}$. In fact, the arguments of [Ma03] show that if $k > \frac{n}{p}$, then any conformal Killing field vanishing at infinity vanishes identically in a neighbourhood of infinity. The blowup arguments used to extend this neighbourhood to the entire manifold require the extra derivative.

Combining [Ma03] Proposition 5.4 and [Ma03] Theorem 6.4 we obtain the following theorem giving conditions under which system (16) is solvable.

[Ma03] Theorem 5.6. *Suppose (M,g) is asymptotically Euclidean of class $W^{k,p}_\rho$ with $k \geq 2$ and suppose $2-n<\delta<0$. If either $k>n/p+1$ or $k>n/p$ and (M,g) has no nontrivial conformal Killing fields X in $W^{k,p}_\delta(M)$ satisfying $BX=0$, then there exists a unique solution $X \in W^{k,p}_\delta(M)$ of (16).*

3.6 Solving the BVP

At this final stage of the construction we have an asymptotically Euclidean manifold (M,g) with $\lambda_g>0$, $R=0$, $h<0$, and a transverse traceless tensor σ with $(n-1)h \leq \sigma(\nu,\nu) \leq 0$. To solve the constraint equations with apparent horizon boundary condition, it suffices to find a conformal factor ϕ solving

$$\begin{aligned} -\Delta\,\phi - \frac{1}{a}\,|\sigma|^2\,\phi^{-3-2\kappa} &= 0 \\ \partial_\nu\phi + \frac{1}{\kappa}h\phi - \frac{2}{a}\sigma(\nu,\nu)\phi^{-1-\kappa} &= 0 \quad \text{on } \partial M. \end{aligned} \tag{18}$$

The following theorem shows that we can solve this system given our choice of (M,g) and σ.

[Ma03] Theorem 4.3. *Suppose (M,g) is asymptotically Euclidean of class $W^{k,p}_\delta$, $k \geq 2$, $k>n/p$, and $2-n<\delta<0$. Suppose also that $\lambda_g>0$, $R=0$, and $h \leq 0$. If $\sigma \in W^{k-1,p}_{\delta-1}$ is a transverse traceless tensor on M such that $(n-1)h \leq \sigma(\nu,\nu) \leq 0$ on ∂M, then there exists a conformal factor ϕ solving (18). Moreover, setting $\hat{g}=\phi^{2\kappa}g$ and $\hat{K}=\phi^{-2}\sigma$, we have that $(M,\hat{g})$ is asymptotically Euclidean of class $W^{k,p}_\delta$, $\hat{K} \in W^{k-1,p}_{\delta-1}$, $(M,\hat{g},\hat{K})$ solves the Einstein constraint equations with apparent horizon boundary condition, and ∂M is a marginally trapped surface.*

Setting $\phi = 1+v$, the problem reduces to finding a solution of

$$\begin{aligned} -\Delta\,v &= \frac{1}{a}\,|\sigma|^2\,(1+v)^{-3-2\kappa} \\ \partial_\nu v &= -\frac{1}{\kappa}h(1+v) + \sigma'(1+v)^{-1-\kappa} \quad \text{on } \partial M, \end{aligned} \tag{19}$$

where $\sigma' = \frac{2}{a}\sigma(\nu,\nu)$. The method of proof is to find a supersolution v_+ and a subsolution v_- of this system such that $v_- \leq v_+$. Recall that a subsolution satisfies

$$\begin{aligned} -\Delta\,v_- &\leq \frac{1}{a}\,|\sigma|^2\,(1+v_-)^{-3-2\kappa} \\ \partial_\nu v_- &\leq -\frac{1}{\kappa}h(1+v_-) + \sigma'(1+v_-)^{-1-\kappa} \quad \text{on } \partial M, \end{aligned}$$

and a supersolution is defined similarly with the inequalities reversed. Now the hypotheses $(n-1)h \leq \sigma(\nu,\nu)$ immediately implies $v_-=0$ is a subsolution. So all the work is contained in finding a *non-negative* supersolution. We define v_+ to be the solution of

$$\begin{aligned}-\Delta\, v_+ &= \frac{1}{a}\,|\sigma|^2\\ \partial_\nu v_+ + \frac{1}{\kappa} h v_+ &= -\frac{1}{\kappa} h.\end{aligned}$$

This solution exists since $\lambda_g > 0$. However, since $h < 0$, we can't use a maximum principle to show that $v_+ \geq 0$. The proof of this fact comes from considering a parameterized family of equations and using a weak Harnack inequality in an argument similar to that used in [Ma03] Proposition 4.1.

The existence of a solution is then deduced from the following general existence theorem for the system

$$\begin{aligned}-\Delta\, u &= F(x,u)\\ \partial_\nu u &= f(x,u) \quad \text{on } \partial M.\end{aligned} \tag{20}$$

Proposition 3.2. *Suppose (M, g) is asymptotically Euclidean of class $W^{2,p}_\rho$ with $p > n/2$. Suppose also that $2 - n < \delta < 0$ and that $u_-,\ u_+ \in W^{2,p}_\delta$ are a subsolution and a supersolution respectively of (20) that satisfy $u_- \leq u_+$. Let $D = \{(x, y) \in M \times \mathbb{R} : u_-(x) \leq y \leq u_+(x)\}$, let $D_\partial = (\partial M \times \mathbb{R}) \cap D$, and let $\mathcal{D} = \{v \in W^{2,p}_\delta : \text{graph } v \subset D\}$. Suppose that*

1. *$F(x, y)$ is continuous in y on D and $\partial_y F(x, y) + V(x) > 0$ on D for some non-negative $V \in L^p_{\rho-2}$,*
2. *$||F(x, v)||_{L^p_{\delta-2}(M)} \lesssim 1$ for all $v \in \mathcal{D}$,*
3. *$f(x, y)$ is continuous in y on D_∂ and $\partial_y f(x, y) + \mu(x) > 0$ on D_∂ for some non-negative $\mu \in W^{1-\frac{1}{p},p}$.*
4. *$||f(x, v)||_{W^{1-\frac{1}{p},p}(\partial M)} \lesssim ||v||_{W^{1-\frac{1}{q},q}(\partial M)} + 1$ for all $v \in \mathcal{D}$ and for some $q < p^*$.*

Then there exists a solution $u \in \mathcal{D}$ of (20).

4. Ongoing Research

4.1 Deeper analysis of the boundary value problem

Although we have found sufficient conditions on (M, g, σ) to allow for the construction of solutions of

$$\begin{aligned}-\Delta\,\phi + \frac{1}{a}\Big(R\phi - |\sigma|^2\,\phi^{-3-2\kappa}\Big) &= 0\\ \frac{a}{2}\phi^{-1-\kappa}\partial_\nu\phi + \frac{1}{\kappa}h\phi^{-\kappa} - \frac{2}{a}\phi^{-2-2\kappa}\sigma(\nu,\nu) &= 0 \quad \text{on } \partial M,\end{aligned} \tag{21}$$

it would be desirable to find all data (M, g, σ) for which (21) is solvable. Ideally, we would like to find a condition $\mathcal{C}(M, g, \sigma)$ allowing for the construction of a unique solution of (21). This would yield a parameterization of CMC solutions of the Einstein constraint equations with apparent horizon boundary condition.

Theorem to be Proved 1. *Suppose (M, g) is asymptotically Euclidean of class $W^{k,p}_\delta$, $k \geq 2$, $k > n/p$, and $2 - n < \delta < 0$ and that $\sigma \in W^{k-1,p}_{\delta-1}$ is a transverse traceless tensor. There exists a solution ϕ of (21) with $1 - \phi \in W^{k,p}_\delta$ if and only if $\mathcal{C}(M, g, \sigma)$ holds, and any such solution is unique.*

Now if (M, g, σ) has a corresponding solution ϕ, then $(M, \psi^{2\kappa} g, \psi^{-2}\sigma)$ has a corresponding solution $\phi\psi^{-1}$. So the condition $\mathcal{C}(M, g, \sigma)$, if it can be found, will be conformally invariant in the sense that if $\mathcal{C}(M, g, \sigma)$ is true, then so is $\mathcal{C}(M, \psi^{2\kappa} g, \psi^{-2}\sigma)$. Hence conformal invariance is an important benchmark for the condition $\mathcal{C}$. We recall that for manifolds without boundary, the corresponding condition $\mathcal{C}'(M, g, \sigma)$ is simply $\lambda_g > 0$. We would like to find a similar, concise, manifestly conformally invariant condition for the boundary value problem. The two conditions of [Ma03] that need to be revisited are $\lambda_g > 0$ and $(n - 1)h \leq \sigma(\nu, \nu)$ in a gauge where $R = 0$ and $h \leq 0$.

The property $\lambda_g > 0$ is conformally invariant, but it is not clear that it is necessary. Given the central role this condition plays in the construction of [Ma03], and the inherent difficulty in working with metrics with $\lambda_g \leq 0$, it is important to establish whether or not $\lambda_g > 0$ is required for solutions to exist. Since there is is not obvious reason why this should hold, a first step in this direction would be to attempt the construction of a counterexample in, say, spherical symmetry.

The condition on the transverse traceless tensor, $(n - 1)h \leq \sigma(\nu, \nu)$, is made with respect to a conformal representative $R = 0$ and $h \leq 0$. There are many such representatives corresponding to different choices of $h \leq 0$. Since the inequality is not conformally invariant, we see it cannot be necessary. A more subtle interaction between the metric and $\sigma(\nu, \nu)$ needs to be found.

Another question arises in connection with the difference between the results of [Ma03] and [Da03]. From these papers, we know we can find solutions of the constraint equations such that either

$$\theta_- \leq \theta_+ = 0 \qquad \text{(implying } h \leq 0\text{)}$$
$$\theta_+ \leq \theta_- \leq 0 \qquad \text{(implying } h \geq 0\text{)}$$

Neither construction allows one to find surfaces with $\theta_- \leq \theta_+ < 0$. We can find one-parameter families of initial data starting with the construction in [Ma03] and terminating with the construction in [Da03], but only by passing through the condition $\theta_+ = \theta_- = h = 0$. We would like to understand better the relationship between the two constructions by determining first if one can construct solutions with $\theta_- \leq \theta_+ < 0$ (i.e. trapped surfaces with $h \leq 0$), and second if we can pass between the two constructions without going through the condition $h = 0$.

Let us consider what happens when we adopt the approach of [Ma03] to find solutions with $\theta_+ < 0$. The boundary value problem in the gauge $R = 0$ and $h < 0$ then becomes

$$-\Delta v = \frac{1}{a}\left|\sigma\right|^2(1+v)^{-3-2\kappa}$$
$$\partial_\nu v + \frac{1}{\kappa}h(1+v) - \frac{2}{a}(1+v)^{-1-\kappa}\sigma(\nu,\nu) = -\frac{2}{a}\theta_+(1+v)^{1+\kappa} \quad \text{on } \partial M.$$

The difficulty is that the right hand side of the boundary condition has the wrong sign for finding a supersolution. The function $-\theta_+(1+v)^{1+\kappa}$ is increasing in v when $\theta_+ < 0$, which interferes with finding a positive supersolution. It seems likely that an implicit function theorem argument can be used to perturb off of the case $\theta_- < \theta_+ = 0$ to find solutions with $\theta_- < \theta+ < 0$. However, an implicit function theorem argument would have a smallness condition that is not needed to find solutions with $\theta_+ \leq \theta_- \leq 0$. It is not clear why there should be an asymmetry between the two types of solutions, and it would be nice to find a construction not involving such a smallness condition.

4.2 Very Rough Solutions of the Constraint Equations

Although the results of [Ma03] construct low regularity solutions of the constraint equations, the choice of function spaces is not quite optimal. Traditional constructions for solutions of the constraint equations work with functions that are either locally in a Hölder space or a Sobolev space $W^{k,p}$ with k an integer. On the other hand, existence theorems for the evolution problem typically require initial data in H^s_{loc}. From [Ma03] we see that when $n = 3$ we can construct solutions in $W^{2,p}_\delta$ with $p > 3/2$ and certainly in $W^{2,2}_\delta = H^2_\delta$. Moreover, elliptic regularity arguments show that we can also construct solutions in H^s_δ with $s \geq 2$ and not necessarily an integer. However, it has been conjectured [KR02] that the vacuum Einstein equations are well posed for $(g, K) \in H^{3/2+\epsilon} \times H^{1/2+\epsilon}$. Although there has been doubt expressed about this conjecture [KR03], the elliptic constraint problem is easier to work with than the hyperbolic evolution equations in the low regularity setting. The lower bound $H^{3/2+\epsilon} \times H^{1/2+\epsilon}$ is natural for the constraints, so we are motivated to find such solutions, both in the asymptotically Euclidean and compact settings.

There are two analytic results required to extend the construction of [Ma03] to these spaces. First, we must establish the Fredholm properties of the conformal Laplacian of a $H^{3/2+\epsilon}$ metric acting on $H^{3/2+\epsilon}$. The curvature of the metric is only defined as a distribution, but this does not appear to be a major obstacle to proving the Fredholm properties. More seriously, we also require a weak Harnack inequality used to ensure the positivity of conformal factors. Ideally we would like to prove that there exists $C > 0$ such that every nonnegative supersolution $u \in H^{3/2+\epsilon}(B_1)$ of $-\Delta + R$ satisfies

$$||u||_{L^2(B_1)} \leq C \inf_{B_1} u.$$

This is not a standard result, since $R \in H^{-\frac{1}{2}+\epsilon}$ and is not in an L^p space. Moser iteration is the likely tool to obtain the estimate, but there are technical details to be worked out due to the the low regularity.

In the compact setting, there is another missing tool. The usual construction of CMC initial data [Is95] uses the full force of the Yamabe problem. We are not inclined to prove this theorem for low regularity metrics, especially since it seems unappealing that an existence theorem for the constraint equations should require such a deep result. Rather, we should find a construction that avoids conformally transforming to metrics with constant scalar curvature. Presumably metrics having curvature with constant sign should be enough to prove the result.

Now the a priori estimates of [KR02] and [ST] together with an existence theorem for initial data of the correct class are not enough to obtain an existence theorem for the evolution problem. We also need to show that one can approximate very rough solutions of the constraint equations with smooth solutions of the constraint equations. There are two possible approaches to address this need. First, we can show that the specific solutions we construct via the conformal method can be approximated by smooth solutions. It seems promising to approximate the starting data for the conformal method (M, g, σ) with a sequence smooth data (M, g_k, σ_k) and use a priori estimates to obtain convergence of the resulting solutions. It would be better to obtain a general density theorem for low regularity solutions using, for example, implicit function theorem arguments. Regardless of the approach, a proof of the density of smooth solutions should accompany a construction of very rough solutions of the constraints.

4.3 The Extension Problem

We finish by describing an interesting problem concerning the constraint equations not directly related to the work in [Ma03]. There have recently appeared gluing constructions for the constraint equations indicating the malleable nature of solutions of the constraints. We mentioned before the construction of [IMP02] that allows one to construct new solutions from old by means of surgery, such that the new solution is a perturbation of the old one far from the surgery site. Another gluing construction, due to Corvino [Co00], allows one to take asymptotically Euclidean initial data and perturb the solution near infinity to be exactly Schwarzchild. An interesting aspect of this construction is that the solution is only altered in a neighbourhood of infinity, and hence agrees exactly with the original solution away from infinity. Inspired by techniques from [Co00], it has been shown [CD03] that glued solutions created by the techniques of [IMP02] can be perturbed away from the surgery site to agree exactly with the original solutions.

Given the flexible nature of solutions of the constraints, we are lead to reconsider a problem proposed as part of the quasi-local mass programme of Bartnik. A simple version of the problem can be phrased as follows. Given a solution of the constraint equations (B_1, g, K) on the unit ball, smooth up to the boundary, does there exist an asymptotically Euclidean solution (M', g', K') and an embedding $\iota : M \to M'$ such that $\iota^* g' = g$ and $\iota^* K' = K$? In other words, are there any obstructions to (B_1, g, K) being realized as a subset of an asymptotically Euclidean manifold? This is a hard problem. An easier, related problem is to ask for a local extension. Can we find a solution on (B_2, g, K) agreeing with the original solution on B_1? This would be a necessary condition for having a global embedding. Having a local extension might also permit a gluing construction for a global extension, but this is not at all clear. Even the local extension question seems hard, and has several easier questions that can be reduced from it. Given a scalar flat metric on the ball, does it have an extension? Given a metric g and a harmonic function f on the ball, does the pair (g, f) have an extension? Even in this last case, it is not clear what obstructions might exist. Suppose g is the flat metric. Then f is analytic in the interior of B, but f might fail to be analytic at the boundary. In this case, there is no hope of extending g to be the flat metric. But this is not a real obstruction, since the extended metric can fail to be analytic at the boundary also and thereby permit the possibility of an extension. It would be interesting to determine whether such extensions exist and if they can be parameterized in a useful way.

5. References

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