Asymptotic Behavior of Marginally Trapped Tubes

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Outline of Talk

1. Preliminaries
2. Examples of marginally trapped tubes
3. Asymptotic behavior in general?
4. General black hole spacetimes
5. Main result
6. Application to Higgs fields

Preliminaries — general relativity

General relativity says that spacetime is described by a Lorentzian 4-manifold $(\mathcal{M}, g)$ satisfying the Einstein field equations

$$R_{\alpha\beta} - \frac{1}{2} R\, g_{\alpha\beta} = 8\pi T_{\alpha\beta}$$

where

$R_{\alpha\beta}$ is the Ricci curvature of g,

R is the scalar curvature of g, and

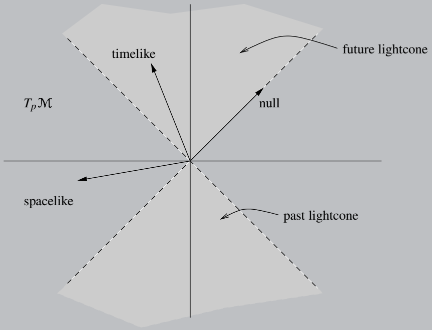
$T_{\alpha\beta}$ is the stress-energy tensor describing all matter and energy in the spacetime.

Preliminaries — causal notions

Since g has signature $(-,+,+,+)$, we can partition tangent vectors of $\mathcal{M}$ into three types: for $X \in T_p\mathcal{M}$,

$$
\begin{aligned}
g(X,X) < 0 \quad &\Longleftrightarrow \quad X \text{ is } \textit{timelike} \\
g(X,X) = 0 \quad &\Longleftrightarrow \quad X \text{ is } \textit{null} \text{ (or } \textit{lightlike}\text{)} \\
g(X,X) > 0 \quad &\Longleftrightarrow \quad X \text{ is } \textit{spacelike};
\end{aligned}
$$

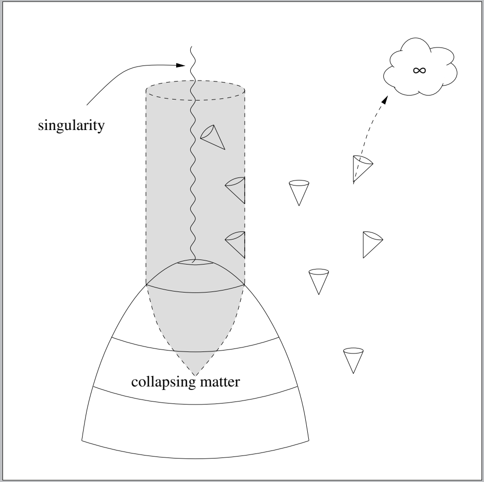
if X is either timelike or null, it is called *causal*;
if X is either null or spacelike, it is *achronal*.



We will assume that $\mathcal{M}$ is *time orientable* — causal vectors may be partitioned into two sets, the future- and past-directed lightcones.

Causal characterizations also extend to differentiable curves and submanifolds.

Preliminaries — black holes (heuristically)

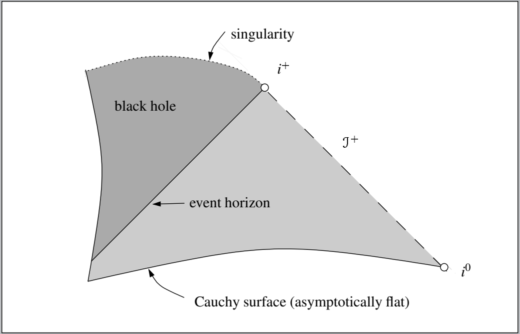


Preliminaries — black holes (mathematically)

To make this idea rigorous, one must locate future null infinity, $\mathcal{I}^+$.

One traditionally does this by making a conformal compactification of the spacetime $(\mathcal{M}, g)$ and identifying $\mathcal{I}^+$ as a null part of its conformal boundary.

The black hole region is then $\mathcal{M} \setminus J^-(\mathcal{I}^+)$.



Note that one must have the entire spacetime at hand in order to find the black hole.

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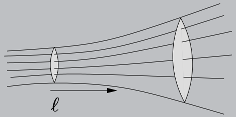
Definition

Given any spacelike 2-surface S and a future null vector field ℓ^α orthogonal to it, the *expansion* of S in the direction ℓ is

$$\theta_{(\ell)} = \mathrm{div}_S\,\ell$$

(the derivatives are taken with respect to the connection on $\mathcal{M}$, the trace with respect to the induced Riemannian metric on S).

The expansion $\theta_{(\ell)}$ measures the infinitesimal change in surface area of S in the direction ℓ.

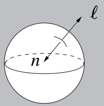


Preliminaries — trapped surfaces

A spacelike 2-surface S has exactly two orthogonal future null directions, given by vector fields ℓ and n, say.

If ℓ points "out" and n points "in," set

$$\begin{aligned} \theta_+ &= \theta_{(\ell)} \\ \theta_- &= \theta_{(n)}. \end{aligned}$$



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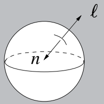

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If $\theta_- < 0$ and $\theta_+ < 0$, the surface S is said to be *trapped*.

If $\theta_- < 0$ and $\theta_+ = 0$, then S is said to be *marginally trapped*.

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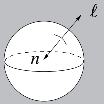

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Singularity Theorem (Penrose, 1965)

Let $(\mathcal{M}, g)$ be a connected, globally hyperbolic spacetime whose Cauchy surface is noncompact and which satisfies the null energy condition. If $\mathcal{M}$ contains a closed trapped surface S, then $\mathcal{M}$ is singular — that is, it contains at least one inextendible future-directed null geodesic emanating from S and having finite affine length in $\mathcal{M}$.

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Related terminology:

- a *dynamical horizon* is an MTT which is spacelike;
- an *isolated horizon* is (essentially) an MTT which is null;
- a *timelike membrane* is an MTT which is timelike.

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Dynamical and isolated horizons appear to be well-suited to model the surfaces of dynamical and equilibrium black holes, respectively [Ashtekar & Krishnan, others].

Regardless of whether these definitions should replace the traditional one for black holes, understanding the behavior of marginally trapped tubes is a step toward understanding black holes' interiors.

A spherically symmetric spacetime is one which admits an $SO(3)$-action by isometries. One can work with the 1+1-dimensional Lorentzian quotient manifold

$$\mathcal{Q} = \mathcal{M}/SO(3)$$

instead of $\mathcal{M}$ without loss of information.

Conformally embedding $\mathcal{Q}$ into a bounded subset of $\mathbb{M}^{1+1}$, we obtain a *Penrose diagram*, off of which essentially all causal and asymptotic information may be read.

Preliminaries — spherical symmetry

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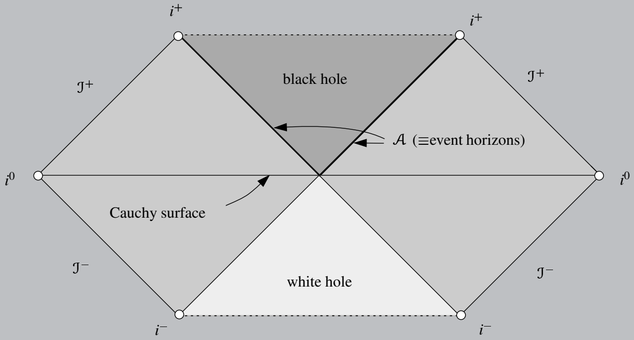
Remark. All known analytical (exact) examples of MTTs are spherically symmetric, and all existing analytical theorems concerning their asymptotic behavior assume spherical symmetry.

The marginally trapped tubes in a Schwarzschild spacetime of mass M are isolated horizons which coincide with the black hole event horizons.



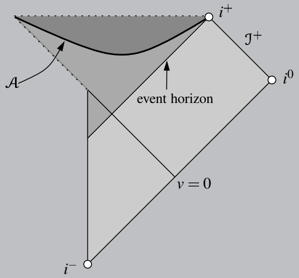
The same is true in a Reissner-Nordström (electrovac) spacetime of mass M and charge e.

Vaidya spacetimes with nonconstant, nondecreasing mass functions $M(v)$ provide the simplest examples of dynamical horizons.

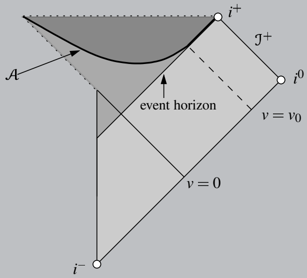
$$g = -\left(1 - \frac{2M(v)}{r}\right) dv^2 + 2dvdr + r^2 g_{S^2},$$

$$T = \frac{\dot{M}(v)}{r^2} dv^2,$$

where and $M(v)$ is any smooth function of v.



$M(v) \nearrow M_0$ as $v \to \infty$



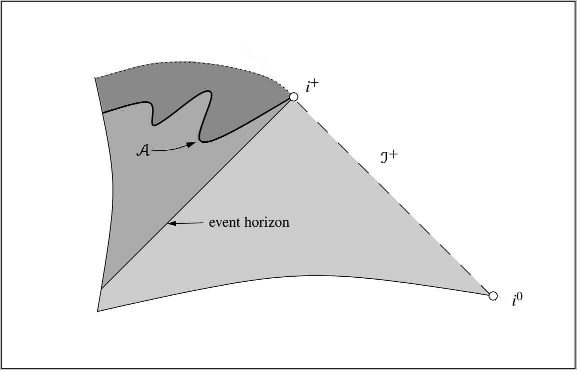
$M(v) \equiv M_0$ for $v \geq v_0$

Examples — evolutionary setting

For certain matter models, it is known that the maximal development of spherically symmetric asymptotically flat initial data contains an MTT which is asymptotic to the event horizon (i.e. terminates at i^+):

* massless scalar fields [Christodoulou 1993]
* Einstein-Maxwell scalar fields [Dafermos 2005, Dafermos & Rodnianski 2005]
* Einstein-Vlasov (collisionless matter) [Dafermos & Rendall 2007]

Additionally, in the first two cases the MTT is known to be achronal near i^+.



Possible Asymptotic Behavior — good

What about the general case, for arbitrary (spherically symmetric) matter?

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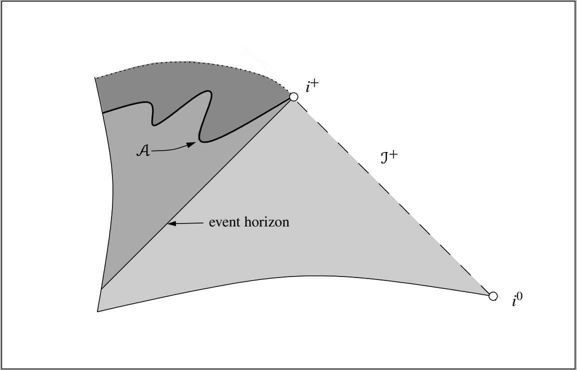
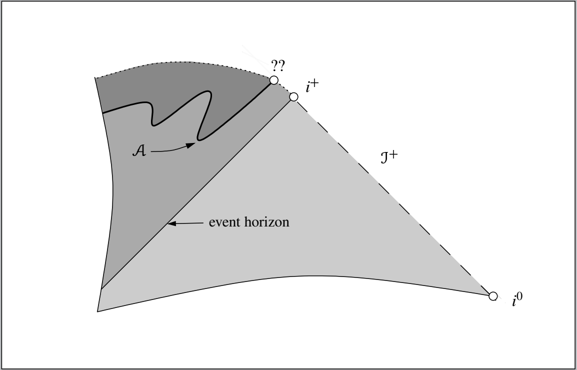
What about the general case, for arbitrary (spherically symmetric) matter?
Must a marginally trapped tube be asymptotic to the event horizon?



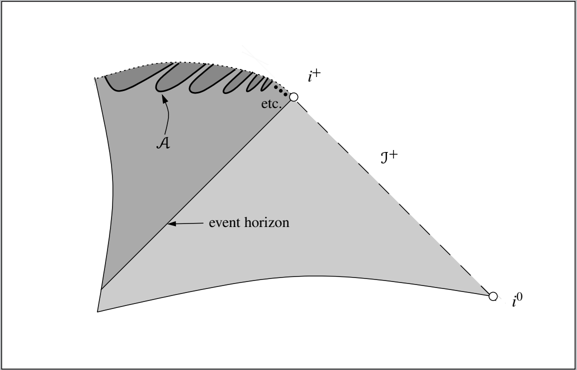
Possible Asymptotic Behavior — bad

Or could it terminate far inside the black hole?



Possible Asymptotic Behavior — ugly

One can imagine all sorts of bad behavior.



General black hole spacetimes — spherical symmetry revisited

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Then conformally embedding $\mathcal{Q} = \mathcal{M}/SO(3) \hookrightarrow (\mathbb{R}^2, -du\,dv)$, its metric takes the form $-\Omega^2 du\,dv$, where $\Omega = \Omega(u,v) > 0$ is smooth on $\mathcal{Q}$. The metric g on $\mathcal{M}$ may be expressed

$$g = -\Omega^2 du\,dv + r^2 g_{S^2},$$

where the radial function $r = r(u,v) \geq 0$ is smooth on $\mathcal{Q}$ and positive away from the center of symmetry, and g_{S^2} is the round metric on S^2.

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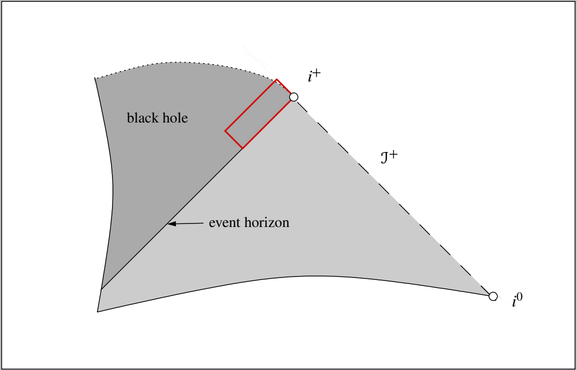
The Einstein equations on $\mathcal{M}$ yields a system of equations on $\mathcal{Q}$:

$$\begin{aligned} \partial_u(\Omega^{-2}\partial_u r) &= -r\Omega^{-2}T_{uu} && (1)\\ \partial_v(\Omega^{-2}\partial_v r) &= -r\Omega^{-2}T_{vv} && (2)\\ \partial_u m &= 2r^2\Omega^{-2}\left(T_{uv}\partial_u r - T_{uu}\partial_v r\right) && (3)\\ \partial_v m &= 2r^2\Omega^{-2}\left(T_{uv}\partial_v r - T_{vv}\partial_u r\right), && (4) \end{aligned}$$

where T_{uu}, T_{uv}, and T_{vv} are component functions of $T_{\alpha\beta}$ on M and m is the Hawking mass,

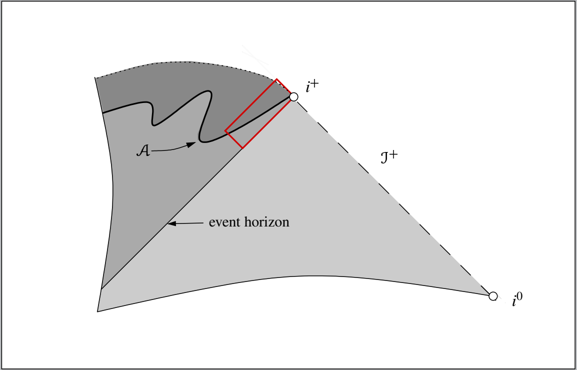
$$m = m(u,v) = \frac{r}{2}\left(1 - |\nabla r|^2\right) = \frac{r}{2}\left(1 + 4\Omega^{-2}\partial_u r\,\partial_v r\right). \tag{5}$$

In order to approach the problem, focus on a characteristic rectangle near timelike infinity, i^+:



General black hole spacetimes — approaching the problem

In order to approach the problem, focus on a characteristic rectangle near timelike infinity, i^+; whether $\mathcal{A}$ is asymptotic to the event horizon or not is determined by its behavior in this rectangle.

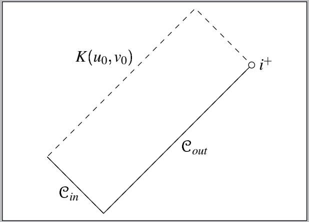


We begin with a characteristic rectangle

$$K = K(u_0, v_0) \; = \; [0, u_0] \times [v_0, \infty)$$

and characteristic initial hypersurfaces

$$\begin{aligned} \mathcal{C}_{\text{in}} \; &= \; [0, u_0] \times \{v_0\} \\ \mathcal{C}_{\text{out}} \; &= \; \{0\} \times [v_0, \infty). \end{aligned}$$



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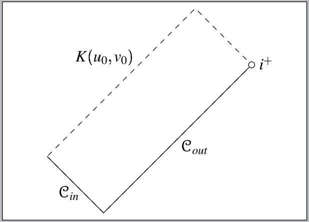

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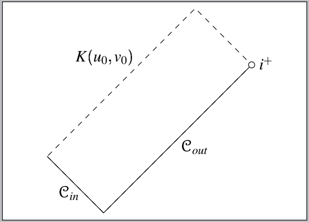

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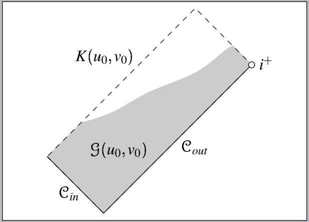

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Finally, assume that we obtain the maximal future development of the system (1)-(5),

$$\mathcal{G}(u_0, v_0) \subset K(u_0, v_0).$$

General black hole spacetimes — trapped surfaces revisited

Each point $(u,v) \in \mathcal{G}(u_0,v_0)$ represents a 2-sphere of radius $r = r(u,v)$ in $\mathcal{M}$.
The two future null normal directions are ∂_u and ∂_v — let u be the ingoing direction and v the outgoing direction. Then:

$$\begin{array}{rcccl} & \theta_+ & = & \theta_{(\partial_v)} & = & 2(\partial_v r)r^{-1} \\ \text{and} & \theta_- & = & \theta_{(\partial_u)} & = & 2(\partial_u r)r^{-1}. \end{array}$$

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Therefore we define the *regular region* as

$$\mathcal{R} = \{(u, v) \in \mathcal{G}(u_0, v_0) : \partial_v r > 0 \text{ and } \partial_u r < 0\},$$

the *trapped region* as

$$\mathcal{T} = \{(u, v) \in \mathcal{G}(u_0, v_0) : \partial_v r < 0 \text{ and } \partial_u r < 0\},$$

and the *marginally trapped tube* as

$$\mathcal{A} = \{(u, v) \in \mathcal{G}(u_0, v_0) : \partial_v r = 0 \text{ and } \partial_u r < 0\}.$$

Note that $\mathcal{A}$ is a hypersurface of $\mathcal{G}(u_0, v_0)$ provided that 0 is a regular value of $\partial_v r$.

We now make a number of additional assumptions on the initial data, its maximal development, and the components of the stress-energy tensor. These assumptions are necessary to insure that matter model is physically reasonable and that the rectangle K is located inside a black hole region as shown previously.

I	$T_{uu} \geq 0,\ T_{uv} \geq 0,$ and $T_{vv} \geq 0$ in $\mathcal{G}(u_0, v_0)$
II	$J^-(\mathcal{G}(u_0, v_0)) \subset \mathcal{G}(u_0, v_0)$
III	$\sup_{\mathcal{C}_{out}} r = r_+ < \infty$
IV	$m \geq 0$ along $\mathcal{C}_{out}$
V	$\partial_u r < 0$ along $\mathcal{C}_{out}$
VI	$\partial_v r > 0$ along $\mathcal{C}_{out}$
VII	'first singularities' in the regular region $\mathcal{R}$ can only arise from the center of symmetry (where $r = 0$).

General black hole spacetimes — assumptions I & II

I $$T_{uu} \geq 0, \;\; T_{uv} \geq 0, \;\; \text{and } T_{vv} \geq 0 \;\; \text{in} \;\; \mathcal{G}(u_0, v_0).$$

These inequalities are what the dominant energy condition on $\mathcal{M}$ boils down to on $\mathcal{Q}$. (Upstairs, the dominant energy condition requires that $-T^{\alpha}{}_{\beta}\xi^{\beta}$ be future-directed causal for all future-directed timelike ξ^{α}.)

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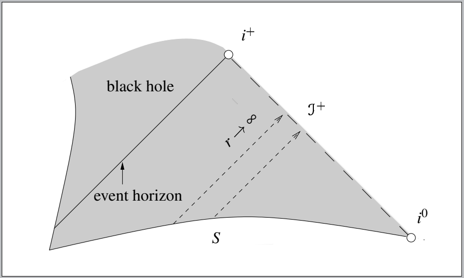
II $$J^{-}(\mathcal{G}(u_0, v_0)) \subset \mathcal{G}(u_0, v_0).$$

This is the statement that the maximal development $\mathcal{G}(u_0, v_0)$ is a past set. (It would come along for free given a global existence result in an evolutionary setting.)

General black hole spacetimes — assumptions III & IV

III
$$\sup_{\mathcal{C}_{out}} r = r_+ < \infty.$$



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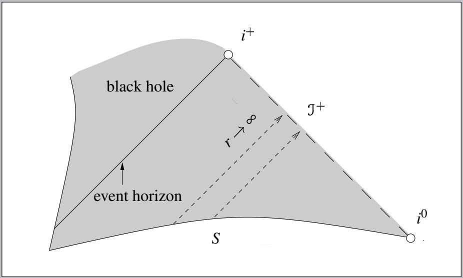


IV

$$m \geq 0 \text{ along } \mathcal{C}_{out}.$$

This is another form of the physically reasonable requirement that energy be locally nonnegative.

General black hole spacetimes — assumption V

V $$\partial_u r < 0 \text{ along } \mathcal{C}_{out}.$$

An *anti-trapped surface* is one for which $\partial_u r \geq 0$, so this is just the statement that no anti-trapped surfaces are present initially. It is motivated primarily by the following:

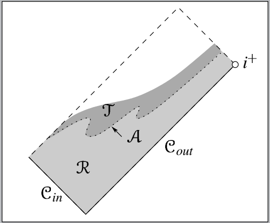
Proposition (Christodoulou)

If $\partial_u r < 0$ along $\mathcal{C}_{out}$, then $\mathcal{G}(u_0, v_0) = \mathcal{R} \cup \mathcal{T} \cup \mathcal{A}$ — that is, anti-trapped surfaces cannot evolve if none are present initially.

General black hole spacetimes — assumption VI

Proposition (Christodoulou)

If $(u,v) \in \mathcal{T} \cup \mathcal{A}$, then $(u,v^*) \in \mathcal{T} \cup \mathcal{A}$ for all $v^* > v$. Similarly, if $(u,v) \in \mathcal{T}$, then $(u,v^*) \in \mathcal{T}$ for all $v^* > v$.



The trapped region $\mathcal{T}$ must be contained inside the black hole, so $\partial_v r \geq 0$ along $\mathcal{C}_{out}$ if the latter is to lie along the event horizon. But if $\partial_v r = 0$ at a single point along $\mathcal{C}_{out}$, then $\mathcal{A}$ must coincide with $\mathcal{C}_{out}$ to the future of that point by the above proposition. Therefore, in order to avoid the trivial case,

VI $$\partial_v r > 0 \text{ along } \mathcal{C}_{out}.$$

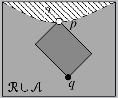
Finally, we need to make use of a certain extension principle. This is known to hold for self-gravitating Higgs' fields and self-gravitating collisionless matter and expected to hold for other physically reasonable matter models [Dafermos 2005, Dafermos & Rendall 2005].

Let

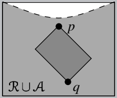
$$\Gamma = \{p \in \mathcal{G}(u_0, v_0) \,:\, r(p) = 0\} \quad \text{(the center of symmetry)}$$

and regard set closures as being taken with respect to the topology of $K(u_0, v_0)$. Then the extension principle may be formulated:

VII If $p \in \overline{\mathcal{R}} \backslash \overline{\Gamma}$, and $q \in \overline{\mathcal{R}} \cap I^-(p)$ such that $J^-(p) \cap J^+(q) \backslash \{p\} \subset \mathcal{R} \cup \mathcal{A}$,



then $p \in \mathcal{R} \cup \mathcal{A}$.



General black hole spacetimes — statement of the problem

Problem

Consider the class of spacetimes $\mathcal{G}(u_0, v_0)$ obtained as described and satisfying assumptions I-VII. Are there general conditions which can be imposed on some or all of r, m, Ω, T_{uu}, T_{uv}, T_{vv} and their derivatives in $\mathcal{G}(u_0, v_0)$ that are sufficient to guarantee that the spacetime will contain a marginally trapped tube $\mathcal{A}$ which is asymptotic to the event horizon?

Main result — statement

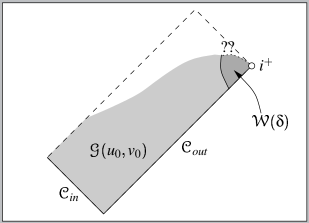
Theorem 1

Suppose $(\mathcal{G}(u_0,v_0),\Omega,r)$ satisfies assumptions I-VII. If there exist a constant $0 < c_0 < \frac{1}{4r_+^2}$, constants $c_1, c_2 > 0$, constants $0 < \varepsilon < \frac{1}{4r_+^2} - c_0$ and $v' \geq v_0$, and some small $\delta > 0$ such that for $\mathcal{W} = \mathcal{W}(\delta) = \{(u,v) : r(u,v) \geq r_+ - \delta\}$ the following conditions hold:

- **A′** $T_{uv}\Omega^{-2} \leq c_0$ in $\mathcal{W}$;
- **B1** $T_{uu}/(\partial_u r)^2 \leq c_1$ in $\mathcal{W} \cap \mathcal{R}$;
- **B2** $\partial_v(\Omega^{-2}T_{uv})(u,\cdot) \in L^1([v_0,\infty))$ for all $u \in [0,u_0]$, and $\int_{v'}^{v} \partial_v(\Omega^{-2}T_{uv})(u,\tilde{v})\,d\tilde{v} < \varepsilon$ for all $(u,v) \in \mathcal{W} \cap \mathcal{R}$ with $v \geq v'$;
- **C** $(-\partial_u r)\Omega^{-2} \leq c_2$ along $\mathcal{C}_{out} \cap \mathcal{W}$,

then the spacetime $\mathcal{G}(u_0,v_0)$ contains a marginally trapped tube $\mathcal{A}$ which is asymptotic to the event horizon. Furthermore, for large v, $\mathcal{A}$ is connected and achronal with no ingoing null segments.

Remarks. $\mathcal{W}$ is essentially a δ-neighborhood of the point i^+:



Also, the expression $T_{uv}\Omega^{-2}$ (seen in conditions $\mathbf{A}'$ and **B2**) takes a particularly simple form in many matter models. For a perfect fluid of pressure P and energy density ρ, it is the quantity $\frac{1}{4}(\rho - P)$. For a self-gravitating Higgs field ϕ with potential $V(\phi)$, it is $\frac{1}{2}V(\phi)$. And for an Einstein-Maxwell massless scalar field of charge e, it is $\frac{1}{4}e^2 r^{-4}$.

Main result — sketch of proof

Lemma 1

If $\mathcal{A}$ is nonempty and $T_{uv}\,\Omega^{-2} < \frac{1}{4r_+^2}$ (condition **A**) holds in $\mathcal{A}$, then each of its connected components is achronal with no ingoing null segments.

Main result — sketch of proof

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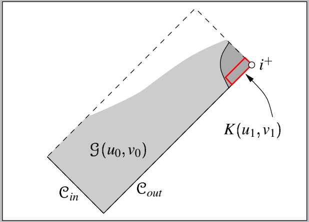
Lemma 2

Suppose condition **A** is satisfied in $\mathcal{W}$. If $\mathcal{G}(u_0, v_0)$ does not contain a marginally trapped tube which is asymptotic to the event horizon, then $\mathcal{W} \cap \mathcal{R}$ contains a rectangle $K(u_1, v_1)$ for some $u_1 \in (0, u_0]$, $v_1 \in [v_0, \infty)$.



Main result — sketch of proof (cont'd)

Rearranging the Einstein equations yields

$$\partial_v \log(-\partial_u r) = 2\kappa r^{-2}\alpha,$$

where

$$\begin{aligned}\kappa &= -\tfrac{1}{4}\Omega^2(\partial_u r)^{-1},\\ \alpha &= m - 2r^3\,\Omega^{-2}T_{uv}\end{aligned}$$

Main result — sketch of proof (cont'd)

Rearranging the Einstein equations yields

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Inside of $K(u_1, v_1)$, we have

Conditions **B1** and **C** $\longrightarrow$ $\kappa \geq \kappa_0 > 0$

Condition **A′** $\longrightarrow$ $\exists$ a small ingoing segment $[0,U]\times\{V\}$ on which $\alpha > \alpha_0 > 0$

Condition **B2** $\longrightarrow$ $\alpha > \alpha_1 > 0$ on $K(U,V) = [0,U]\times[V,\infty)$.

Thus:

$$\partial_v \log(-\partial_u r) > 2\kappa_0 r_+^{-2}\alpha_1.$$

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Integrating along an outgoing ray $\{u\} \times [V, v]$ yields

$$-\partial_u r(u,v) > -\partial_u r(u,V)\, e^{2\kappa_0 r_+^{-2}\alpha_1 (v-V)}.$$

Assume $\partial_u r(u,V) \leq -c < 0$ for all $0 \leq u \leq U$, so that

$$-\partial_u r(u,v) > c e^{2\kappa_0 r_+^{-2}\alpha_1 (v-V)}.$$

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$$-\partial_u r(u,v) > ce^{2\kappa_0 r_+^{-2}\alpha_1(v-V)}.$$

Finally, integrate along an ingoing null ray $[0,u] \times \{v\}$ to get

$$r(u,v) < r(0,v) - ce^{2\kappa_0 r_+^{-2}\alpha_1(v-V)}u.$$

But for any $u > 0$, the RHS tends to $-\infty$ as $v \to \infty$, while the LHS is positive — contradiction.

$\square$

Application to Higgs fields

A self-gravitating Higgs field with non-zero potential consists of a scalar function ϕ on the spacetime and a potential function $V(\phi)$ such that

$$\Box\phi = V'(\phi). \tag{6}$$

The stress-energy tensor then takes the form

$$T_{\alpha\beta} = \phi_{;\alpha}\phi_{;\beta} - \left(\tfrac{1}{2}\phi_{;\gamma}\phi^{;\gamma} + V(\phi)\right)g_{\alpha\beta}. \tag{7}$$

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In spherically symmetry, $\phi = \phi(u,v)$ and the evolution equation (6) becomes

$$V'(\phi) = -4\Omega^{-2}(\partial^2_{uv}\phi + \partial_u\phi\,(\partial_v \log r) + \partial_v\phi\,(\partial_u \log r))$$

and in double-null coordinates, (7) yields

$$\begin{aligned} T_{uu} &= (\partial_u\phi)^2, \\ T_{vv} &= (\partial_v\phi)^2, \text{ and} \\ T_{uv} &= \tfrac{1}{2}\Omega^2 V(\phi). \end{aligned}$$

Note that the dominant energy condition (I) is satisfied if and only if $V(\phi) \geq 0$.
The extension principle (VII) is known to hold for self-gravitating Higgs fields precisely when V is bounded below [Dafermos 2005].

Application to Higgs fields – explicit decay

Theorem 2

Assume we have initial data for the spherically symmetric Einstein-Higgs system satisfying III-VI and for which $V \geq 0$. Fix a constant $p > \frac{1}{2}$ and a function $\eta(v) > 0$ such that $\eta(v)$ decreases monotonically to 0 as v tends to infinity. If V'' is bounded, and if along $\mathcal{C}_{out}$ the initial data satisfy

$$\partial_v r < \eta(v),$$

$$|\partial_v \phi| = O(v^{-p}),$$

$$|V'(\phi)| = O(v^{-p}),$$

$$-(\partial_u r)\Omega^{-2} \text{ is bounded above and away from } 0,$$

and

$$\liminf_{v \to \infty} V(\phi) < \frac{1}{4r_+^2},$$

then the result of Theorem 1 holds for the maximal development of these initial data.

Theorem 3

Assume we have initial data for the spherically symmetric Einstein-Higgs system satisfying III-VI and for which $V \geq 0$. If V'' is bounded and nonnegative, and along $\mathcal{C}_{out}$

$$\partial_v r, |\partial_v \phi|, \text{ and } V(\phi) \text{ are sufficiently small,}$$

$$-(\partial_u r)\Omega^{-2} \text{ is bounded above and away from 0,}$$

$$V'(\phi) < C|\partial_v \phi|,$$

$$\partial_v \phi,\ \partial_u \phi < 0,$$

$$\text{either } V'(\phi) \leq 0 \text{ or } |\inf_{\mathcal{C}_{out}} \phi| < \infty,$$

$$\&\quad \text{a technical inequality relating } V'(\phi), \partial_v \log r, \text{ and } \partial_v \phi \text{ is satisfied,}$$

then the result of Theorem 1 holds for the maximal development of these initial data.

Open questions

- Is smallness for Higgs field initial data on an asymptotically flat (spherically symmetric) spacelike 3-manifold sufficient to obtain a black hole spacetime containing a characteristic rectangle satisfying the conditions of Theorem 1?

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- Perturb away from spherical symmetry … ?